Hodge theory and algebraic cycles

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Abstract

This thesis tackles different problems related to the connection between geometric and Hodge theoretic aspects of algebraic varieties.

One of the main results, joint with Stefan Schreieder and Remy van Dobben de Bruyn, concerns the construction problem for Hodge numbers. We realize all Hodge diamonds in $\mathbb{Z}/m$ for arbitrary $m \geq 2$ by smooth complex projective varieties. This results in a full answer to a question by Kollár about universal polynomial relations between Hodge numbers. Then we investigate the case of positive characteristic, where Hodge symmetry may fail. In this setting, we are able to realize even all asymmetric Hodge diamonds in $\mathbb{Z}/m$. Therefore, we completely understand polynomial relations between Hodge numbers in arbitrary characteristic.

Another main result of this thesis solves the first instances of a conjecture by Griffiths and Harris from 1985 about the degree of curves on very general hypersurfaces. Specifically, a very general complex hypersurface in $\mathbb{P}^4$ of degree $d \geq 6$ is conjectured to contain only curves of degree divisible by d. Based on a degeneration technique developed by Kollár in 1991, we prove this conjecture and its higher-dimensional generalizations for infinitely many values of d. The conjecture by Griffiths and Harris was not known for any d previously. Using the link between this problem and the failure of the integral Hodge conjecture, our result shows that the cokernel of the cycle class map is precisely $\mathbb{Z}/d$ for these hypersurfaces.

In the last part of the thesis, we consider another counterexample to the integral Hodge conjecture, namely the first unirational fourfold with a non-algebraic Hodge class, recently found by Stefan Schreieder. We construct a smooth resolution of Schreieder's conic bundle and study a certain unramified cohomology class on it through a geometric description of the norm residue map in Borel–Moore homology. Our explicit approach allows to get a better understanding of this example and might help to decide in the future whether the constructed non-algebraic class is torsion.

Keywords: Hodge numbers, integral Hodge conjecture, unramified cohomology

Contents

1. Introduction

With the exception of chapter 5, the present thesis is based on the publications [PS19] (see chapter 2), [vDdBP20] (see chapter 3), and [Pau22] (see chapter 4). The individual chapters can be read independently, but all results follow a main theme: The relation between certain *geometrical/algebraic* and *topological/Hodge theoretic* aspects of algebraic varieties.

The *geometry* of an algebraic variety X includes for example the study of its algebraic subvarieties. By considering formal $\mathbb{Z}$-linear sums of these subvarieties up to rational equivalence, one obtains the Chow groups $\mathrm{CH}^\bullet(X)$. These groups may be regarded as an algebraic version of cohomology, but are quite hard to understand and in general not finitely generated.

An important tool to understand the *topology* of X are the cohomology groups $H^\bullet(X, A)$ of the underlying analytic space, for example with coefficients $A = \mathbb{Z}$. If X is a smooth projective $\mathbb{C}$-variety, Hodge theory allows to decompose $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C}$ into finer invariants, the Hodge groups $H^{p,q}(X)$. These $\mathbb{C}$-linear subspaces of $H^k(X, \mathbb{C})$ are not purely topological, but also depend on the structure of X as a compact Kähler manifold.

In chapters 2 and 3, we contribute to the classification of possible Hodge diamonds. This was joint work with Stefan Schreieder in characteristic zero and with Remy van Dobben de Bruyn in positive characteristic. We construct smooth projective varieties in any dimension whose Hodge numbers have arbitrary residues modulo an arbitrary non-zero integer. The residues need to satisfy only the usual restrictions coming from Serre duality and Hodge symmetry. The latter restriction can even be dropped in positive characteristic, so we obtain the strongest possible results in all characteristics.

While the contents of chapters 2 and 3 seem to deal mainly with Hodge theoretic aspects of algebraic varieties, connections to their geometry play a role as well. For example, we prove that there are no unexpected polynomial expressions in the Hodge numbers that are birational invariants. The study of algebraic varieties up to birational equivalence is an inherently geometrical topic. In our proofs, the existence of certain algebraic subvarieties will be crucial to perform blow-up constructions.

In chapter 4, we prove the first instances of a conjecture of Griffiths and Harris. If $X \subset \mathbb{P}^4$ denotes a very general complex hypersurface of degree $d \geq 6$, they conjectured in [GH85] that the degree of every curve $C \subset X$ is divisible by d. This simple sounding question remained open for every single d so far. However, Kollár made significant progress on this problem in [K+91] using degeneration arguments. By carefully combining different approaches in [K+91] through another degeneration, we are able to prove the conjecture for $d = 5005$ and infinitely many further values of d. Moreover, we are able to generalize this result to arbitrary dimensions — of the hypersurface X as well as of the considered subvarieties $C \subset X$.

The conjecture of Griffiths and Harris is strongly linked to the extent by which the integral Hodge conjecture on the hypersurface $X \subset \mathbb{P}^4$ fails. The integral Hodge conjecture, which after Atiyah's and Hirzebruch's counterexample [AH61] can be seen rather as a property of a smooth projective $\mathbb{C}$-variety than as a conjecture, concerns the surjectivity of the cycle class map in a given codimension. This map is a central object in the connection between algebraic and Hodge theoretic aspects of smooth projective varieties over $\mathbb{C}$. It turns out that the conjecture of Griffiths and Harris in degree d is equivalent to the cycle class map in codimension 2 having cokernel $\mathbb{Z}/d$. This is the largest theoretically possible cokernel for a smooth hypersurface $X \subset \mathbb{P}^4$, so the aforementioned result proves that the integral Hodge conjecture fails as much as possible for $d = 5005$ and infinitely many further degrees d.

It already followed from Kollár's work [K+91] that the integral Hodge conjecture for hypersurfaces in $\mathbb{P}^4$ fails in general, and this was in fact the first example where the obstruction to the integral Hodge conjecture did not arise from a torsion class. During the last years, many new counterexamples to the integral Hodge conjecture have been found that satisfy additional properties. For example, Schreieder [Sch19] constructed a unirational fourfold with a non-algebraic integral Hodge class. This is particularly interesting because Voisin [Voi06] proved the integral Hodge conjecture for uniruled (and thus for unirational) threefolds.

In chapter 5, we take a closer look at Schreider's 4-dimensional counterexample. By interpreting unramified cohomology classes in terms of Borel–Moore homology, we obtain a new geometric description of his example. In particular, we are able to describe the algebraic multiple of Schreieder's non-algebraic Hodge class explicitly. Our approach is aimed towards deciding whether the non-algebraic integral Hodge class in his example is actually a torsion class in cohomology. This remains an open question.

1.1. Constructing varieties with prescribed Hodge numbers

Let X be a smooth projective variety over $\mathbb{C}$. In particular, X is a compact Kähler manifold. Therefore, Hodge theory provides a decomposition of the k-th Betti cohomology of X into its (p, q)-pieces for all $0 \le k \le 2n$:

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p+q=k \\ 0 \le p,q \le n}} H^{p,q}(X) , \quad \overline{H^{p,q}(X)} = H^{q,p}(X) .$$

The $\mathbb{C}$-linear subspaces $H^{p,q}(X)$ of $H^k(X, \mathbb{C})$ are generated by forms of type (p, q) in de Rham cohomology. They are naturally isomorphic to the Dolbeault cohomology groups $H^q(X, \Omega_X^p)$.

The dimensions $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$ are called *Hodge numbers* and are important numerical invariants of X. While the Betti numbers $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C}) = \sum_{p+q=k} h^{p,q}(X)$ are purely topological invariants, the Hodge numbers $h^{p,q}(X)$ also encode information which depends on the structure of X as a complex manifold. For example, $h^{1,1}(X) = 1$ would tell us that $\operatorname{Pic} X \cong \mathbb{Z}$ (up to torsion), and $h^{p,0}(X) \neq 0$ would indicate the presence of non-zero global holomorphic p-forms.

It is convenient to arrange the Hodge numbers of X in the following way:

$$\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & \iddots & & \vdots & & \ddots & & \\
 & h^{n,1} & & & & & & h^{1,n} & \\
h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n} \\
 & h^{n-1,0} & & & & & & h^{0,n-1} & \\
 & & \ddots & & \vdots & & \iddots & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & &
\end{array}$$

This collection of numbers is called the *Hodge diamond* of X.

Since $H^{q,p}(X)$ is the complex conjugate of $H^{p,q}(X)$ (viewed as $\mathbb{C}$-linear subspaces of $H^{p+q}(X, \mathbb{C}) = H^{p+q}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$), we have $h^{p,q}(X) = h^{q,p}(X)$ for all p, q. In other words, the Hodge diamond is symmetric with respect to the central vertical axis. An interesting consequence of this symmetry is that the Betti number $b_k(X)$ must be even if k is odd. This is an example how Hodge theory influences the topology of algebraic varieties.

Serre duality implies

$$H^{n-p,n-q}(X) = H^{n-q}(X, \Omega_X^{n-p}) \cong H^q(X, \Omega_X^p)^* = H^{p,q}(X)^*$$

and thus $h^{p,q}(X) = h^{n-p,n-q}(X)$ for all p, q. Therefore, the Hodge diamond is invariant under rotation by 180°.

Combining the two symmetries, we see that the Hodge diamond is symmetric with respect to the central horizontal axis. Alternatively, $h^{p,q}(X) = h^{n-q,n-p}(X)$ can be directly deduced from the hard Lefschetz theorem, which states that the $(n-p-q)$-fold cup product with the Kähler class in $H^{1,1}(X)$ yields an isomorphism $H^{p,q}(X) \cong H^{n-q,n-p}(X)$ for all $p+q < n$.

Since this isomorphism factors through $H^{p+1,q+1}(X)$, it follows that $h^{p,q}(X) \leq h^{p+1,q+1}(X)$ for all $p+q<n$. Finally, since X is connected, we have $h^{0,0}(X) = 1$.

To summarize, the Hodge numbers of X are subject to the following conditions:

(1) $h^{0,0}(X) = 1$ (connectedness)
(2) $h^{p,q}(X) = h^{n-p,n-q}(X)$ for all $0 \leq p, q \leq n$ (Serre duality)
(3) $h^{p,q}(X) = h^{q,p}(X)$ for all $0 \leq p, q \leq n$ (Hodge symmetry)
(4) $h^{p,q}(X) \leq h^{p+1,q+1}(X)$ for all $p+q<n$ (Lefschetz inequality)

A natural question is whether these restrictions suffice to describe the set of all Hodge diamonds of smooth projective $\mathbb{C}$-varieties. This question is equivalent to the following construction problem:

Question 1.1. *Let $(h^{p,q})_{0 \leq p,q \leq n}$ be a collection of non-negative integers satisfying the above four conditions. Does there exist a smooth projective $\mathbb{C}$-variety X such that $h^{p,q}(X) = h^{p,q}$ for all $0 \leq p, q \leq n$?*

If this question in its entirety turns out to have a negative answer, one is interested more generally in the classification of all possible Hodge diamonds that can occur among smooth projective $\mathbb{C}$-varieties.

A related open problem is how the behaviour of smooth projective varieties regarding this question differs from the behaviour of arbitrary compact Kähler manifolds. As Simpson points out in his survey [Sim04], very little is known about these types of problems.

So far, the most complete classification results have been obtained in dimensions 2 and 3, see e.g. [Hun89]. For a surface X, the question about its possible Hodge diamonds essentially reduces to understanding three independent invariants, such as the Chern numbers

$$c_1^2(X) = 10 - 8h^{1,0}(X) + 10h^{2,0}(X) - h^{1,1}(X)$$
$$c_2(X) = 2 - 4h^{1,0}(X) + 2h^{2,0}(X) + h^{1,1}(X)$$

together with the first Betti number $b_1(X) = 2 \cdot h^{1,0}(X)$.

In arbitrary dimension, Schreieder [Sch15] made significant progress on Question 1.1. For example, he proves that Question 1.1 has a positive answer if we only consider the Hodge numbers $h^{p,q}(X)$ with $p, q, p - q \neq 0$ and $p, q, p + q \neq n$. Moreover, the Hodge numbers in row k can be almost arbitrary (the only exception is possibly $h^{k/2,k/2}(X)$) for a fixed dimension n and a fixed index $k \in \{0, \ldots, 2n\}$. In particular, the Hodge numbers closer to the center can be much smaller than the ones farther away from the center in a given row. This is surprising because many common examples (such as complete intersections or abelian varieties) do not exhibit this behaviour.

Now let us focus on results concerning the entire Hodge diamond, without ignoring any Hodge numbers. Kotschick and Schreieder [KS13] proved that Serre duality and Hodge symmetry generate all *linear relations* among the Hodge numbers of smooth projective $\mathbb{C}$-varieties of a fixed dimension. Motivated by this result, Kollár raised the question whether this remains true if one allows arbitrary polynomial expressions in the Hodge numbers.

In chapter 2, we answer Kollár's question. In fact, we prove a much stronger result which completely solves the construction problem for Hodge numbers modulo an arbitrary integer m:

Theorem 1.2. *Let $m \geq 2$ be an integer. For any integer $n \geq 1$ and any collection of integers $(h^{p,q})_{0 \leq p,q \leq n}$ such that $h^{0,0} = 1$ and $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ for $0 \leq p, q \leq n$, there exists a smooth projective $\mathbb{C}$-variety X of dimension n such that*

$$h^{p,q}(X) \equiv h^{p,q} \pmod{m}$$

for all $0 \leq p, q \leq n$.

It is not hard to see that this prohibits the existence of any universal polynomial relation among the Hodge numbers, except for the relations induced by Serre duality and Hodge symmetry. Furthermore, Theorem 1.2 answers Question 1.1 from a number theoretic point of view. Since this is the strongest possible result when considering Hodge numbers modulo m, we also see that the behaviour inside the strictly larger class of compact Kähler manifolds does not change in this case.

Unfortunately, without the "modulo m" part, one cannot expect such a nice answer as in Theorem 1.2 anymore. As Schreieder [Sch15] points out, more complicated inequalities than the Lefschetz inequality may restrict the possible Hodge diamonds. For example, $h^{1,1}(X) = 1$ and $h^{2,0}(X) \geq 1$ together imply $h^{2,1}(X) < 12^6 \cdot h^{3,0}(X)$ in dimension 3, see [Sch15, Proposition 28]. This gives a glimpse of the hopeless complexity in classifying all possible Hodge diamonds.

It is well known that the *outer* Hodge numbers, i. e. $h^{p,q}(X)$ with $p \in \{0, n\}$ or $q \in \{0, n\}$, are birational invariants. Kotschick and Schreieder [KS13, Theorem 2] proved that the only linear expressions in the Hodge numbers that are birational invariants are those consisting solely of the outer Hodge numbers. The generalization to polynomial birational invariants is implied by the following result from chapter 2:

Theorem 1.3. *Let $m \geq 2$ be an integer and let X be a smooth projective $\mathbb{C}$-variety of dimension n. For any collection of integers $(h^{p,q})_{1 \leq p,q \leq n-1}$ such that $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ for $0 \leq p, q \leq n$, there exists a smooth projective $\mathbb{C}$-variety $\tilde{X}$ birational to X such that*

$$h^{p,q}(\tilde{X}) \equiv h^{p,q} \pmod{m}$$

for all $1 \leq p, q \leq n - 1$.

This already indicates our strategy for proving Theorem 1.2. First, we get the outer Hodge numbers correct modulo m, and afterwards we adjust the *inner Hodge numbers*, i. e. all remaining Hodge numbers, by birational transformations. For this, we use a sequence of blow-ups along smooth subvarieties.

Over a field k of positive characteristic, we can still define the Hodge numbers of a smooth projective k-variety X to be $h^{p,q}(X) := \dim_k H^q(X, \Omega^p_X)$. Since our constructions are purely algebraic, Theorem 1.2 is still true in positive characteristic. However, one could expect an even stronger statement to be true now. This is because Hodge symmetry turns out to be wrong in positive characteristic. The first incarnation of this fact was found by Serre [Ser58], who constructed a surface S over $\overline{\mathbb{F}_p}$ satisfying $h^{1,0}(S) = 0$ and $h^{0,1}(S) = 1$.

Using Serre's surface S, van Dobben de Bruyn [vDdB21] proved an analogue in positive characteristic of the result from [KS13] about linear relations among Hodge numbers. Interestingly, Serre duality alone (without Hodge symmetry) already generates all linear relations between Hodge numbers in positive characteristic.

Regarding the construction problem modulo m, the strongest possible result in positive characteristic one might hope for is the following:

Theorem 1.4. *Let k be an algebraically closed field of positive characteristic, and let $m \geq 2$ be an integer. For any integer $n \geq 1$ and any collection of integers $(h^{p,q})_{0 \leq p,q \leq n}$ such that $h^{0,0} = 1$ and $h^{p,q} = h^{n-p,n-q}$ for $0 \leq p, q \leq n$, there exists a smooth projective k-variety X of dimension n such that*

$$h^{p,q}(X) \equiv h^{p,q} \pmod{m}$$

for all $0 \leq p, q \leq n$.

In chapter 3, we will prove this theorem. The general structure of the argument is similar to chapter 2 in the sense that the construction problem is first solved for the outer Hodge numbers, and afterwards the inner Hodge numbers are adjusted via a sequence of suitable blow-ups. However, the failure of Hodge symmetry raises new technical difficulties and makes the construction more complex than for $\mathbb{C}$. In order to avoid embedded resolution of singularities, we rely on Maruyama's theory of elementary transformations of vector bundles.

It follows that any polynomial relation among the Hodge numbers in positive characteristic is induced by Serre duality, thus generalizing the result from [vDdB21] about linear relations.

Theorem 1.4 implies in particular that all pairs $(h^{p,q}, h^{q,p})$ of symmetrically arranged Hodge numbers (except for the ones in the middle row, since $h^{p,q} = h^{q,p}$ is a consequence of Serre duality if $p + q = n$) can simultaneously be different. In fact, they can even be incongruent modulo arbitrary integers $m \geq 2$. Surprisingly, the only variety with asymmetric Hodge diamond that is needed in our constructions is Serre's surface S from [Ser58].

In positive characteristic, it is still true that all outer Hodge numbers are birational invariants. However, due to the lack of Hodge symmetry, this is harder to prove than in characteristic zero, see [CR11]. The following statement, proven in chapter 3, is thus optimal:

Theorem 1.5. *Let k be an algebraically closed field of positive characteristic, Let $m \geq 2$ be an integer and let X be a smooth projective k-variety of dimension n. For any collection of integers $(h^{p,q})_{1 \leq p,q \leq n-1}$ such that $h^{p,q} = h^{n-p,n-q}$ for $0 \leq p, q \leq n$, there exists a smooth projective k-variety $\tilde{X}$ birational to X such that*

$$h^{p,q}(\tilde{X}) \equiv h^{p,q} \pmod{m}$$

for all $1 \leq p, q \leq n - 1$.

Again it follows that the only polynomial birational invariants in the Hodge numbers are the ones consisting solely of the outer Hodge numbers.

1.2. Two questions related to the integral Hodge conjecture

Let X be a smooth projective variety over $\mathbb{C}$. For $0 \le k \le n$, let

$$H^{k,k}(X,\mathbb{Z}) = \{\mu \in H^{2k}(X,\mathbb{Z}) \mid \mu_{\mathbb{C}} \in H^{k,k}(X)\}$$

denote the group of *integral Hodge classes* in codimension k, i. e. the integral cohomology classes of degree $2k$ corresponding to a class of type (k,k) in $H^{2k}(X,\mathbb{C})$. In particular, this includes the torsion subgroup of $H^{2k}(X,\mathbb{Z})$.

It is well known that a subvariety $Z \subset X$ of codimension k yields an integral Hodge class $[Z] \in H^{k,k}(X,\mathbb{Z})$. Since rational equivalence of cycles implies homological equivalence, we get a well-defined map

$$\mathrm{CH}^k(X) \to H^{k,k}(X,\mathbb{Z}) \ ,$$

the *cycle class map*. This map plays a central role in the connection between algebraic cycles and Hodge theory.

There has been a lot of work and open conjectures around the kernel and cokernel of this map. Most prominently, the *integral Hodge conjecture* states that the cycle class map is surjective. In codimension $k = 1$, this follows from the Lefschetz $(1,1)$ theorem. For $k = 2$, however, Atiyah and Hirzebruch [AH61] found a 6-dimensional counterexample. In their example, the integral Hodge conjecture fails due to a torsion class. Since then, the integral Hodge conjecture, despite its name, was not studied as an open conjecture anymore, but rather as a property that a smooth projective $\mathbb{C}$-variety might satisfy or not.

The first example where the integral Hodge conjecture fails due to a non-torsion class was found by Kollár [K+91]. It is actually simpler to construct than the original counterexample of Atiyah and Hirzebruch: Kollár considers a very general hypersurface $X \subset \mathbb{P}^4$ of a certain degree d, which we will chose later. The group $H^{2,2}(X,\mathbb{Z})$ is quite easy to understand: By the Lefschetz hyperplane theorem, we have $H^2(X,\mathbb{Z}) = \mathbb{Z}\cdot\alpha$, where $\alpha \in H^2(X,\mathbb{Z})$ denotes the hyperplane class. By Poincaré duality, we conclude that $H^4(X,\mathbb{Z}) = \mathbb{Z}\cdot\frac{1}{d}\alpha^2$, since $\alpha^3 = d$. In particular, we have $H^{2,2}(X,\mathbb{Z}) = H^4(X,\mathbb{Z})$. The integral Hodge conjecture thus asserts that X admits a 1-cycle of degree 1 (where the *degree* of a 1-cycle is given by its pairing with α).

Via a degeneration argument, Kollár was able to show that the degree of every curve $C \subset X$ is divisble by k if we choose $d = k^3$ or $d = 3k^2$ with $\gcd(k,6) = 1$ and $k \ge 5$. A similar example in [K+91], due to van Geemen, shows that the same conclusion remains true for the asymptotically smaller degree $d = 18k$ with $k \ge 9$ and $k \equiv 3$

(mod 6). Results of Debarre, Hulek, and Spandaw about very ample line bundles on Abelian varieties [DHS94] extend this statement to all degrees $d = 6k$ with odd $k \geq 9$.

Perhaps surprisingly, the integral Hodge conjecture was not explicitly mentioned in [K+91]. Instead, the results were motivated by a series of conjectures of Griffiths and Harris [GH85] about curves on very general hypersurfaces $X \subset \mathbb{P}^4$ of degree $d \geq 6$. The weakest of their conjectures states that the degree of every curve $C \subset X$ is divisible by d, which is stronger than any of the above results from [K+91].

The condition $d \geq 6$ is necessary to avoid the existence of lines. Clearly, a very general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$ always contains a curve of degree d, given as a general plane intersection $X \cap \mathbb{P}^2$. In terms of the integral Hodge conjecture, $\alpha^2 \in H^4(X, \mathbb{Z})$ is always algebraic, i. e. every integral Hodge class becomes algebraic after multiplication with d. In particular, the rational Hodge conjecture is true for X.

Let us describe more precisely how (the failure of) the integral Hodge conjecture for X is related to the conjecture of Griffiths and Harris about the possible degrees of curves on X. Let r denote the greatest common divisor of the degrees of all curves $C \subset X$, i. e. the smallest positive degree of a (not necessarily effective) 1-cycle on X. Let $Z^4(X)$ be the cokernel of the cycle class map $\mathrm{CH}_1(X) \to H^{2,2}(X, \mathbb{Z})$. Then we have $Z^4(X) = \mathbb{Z}/r$. The integral Hodge conjecture says that $Z^4(X) = 0$ or $r = 1$. The conjecture of Griffiths and Harris predicts that $r = d$ or $Z^4(X) = \mathbb{Z}/d$, meaning that the integral Hodge conjecture fails to the maximum possible extent.

Apart from the results in [K+91] sketched above, not much is known about the degrees of curves on a very general hypersurface $X \subset \mathbb{P}^4$. It is a result of Wu [Wu90] that every curve $C \subset X$ having degree $\leq 2d - 2$ is a complete intersection with a surface in $\mathbb{P}^4$. In particular, every curve $C \subset X$ has degree $\geq d$. However, this does not restrict the divisibility of $\deg C$ and hence not the possible degrees of 1-cycles on X. For example, there might exist two curves on X whose degrees are very large, but differ only by 1.

The main result of chapter 4 is the following:

Theorem 1.6. *There exist infinitely many degrees d where the conjecture of Griffiths and Harris is true.*

Recall that the conjecture was not known for a single d before. The degrees appearing in Theorem 1.6 are described by an explicit number-theoretic condition and have positive density among the natural numbers. The smallest of them is $d = 5005 = 5 \cdot 7 \cdot 11 \cdot 13$.

For an even larger set of degrees d, having density 1, we are able to disprove the integral Hodge conjecture. Moreover, we construct a smooth projective hypersurface $X \subset \mathbb{P}^4$ defined over $\mathbb{Q}$ such that the degree of every curve $C \subset X$ is divisible by the degree of X (which is not 1). This is based on a construction by Totaro [Tot13] producing examples over $\mathbb{Q}$ for the results from [K$^+$91].

Finally, Theorem 1.6 generalizes from threefolds to hypersurfaces of arbitrary dimensions, and from 1-cycles to arbitrary positive-dimensional cycles:

Theorem 1.7. *For every $n \geq 3$, there exists a set of degrees d with positive density such that the degree of every positive-dimensional algebraic cycle on a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d is divisible by d.*

Let us come back to the integral Hodge conjecture in general. As we have seen, very general hypersurfaces $X \subset \mathbb{P}^4$ of suitably chosen degrees $d \geq 6$ provide counterexamples to the integral Hodge conjecture. Since these threefolds are of general type, one might ask whether there exist 3-dimensional counterexamples of Kodaira dimension < 3.

Voisin [Voi06] showed that no such examples of Kodaira dimension $-\infty$ exist. In other words, the integral Hodge conjecture is true for uniruled threefolds. Totaro [Tot21] proved the integral Hodge conjecture for threefolds X of Kodaira dimension 0 with $H^0(X, \omega_X) \neq 0$. On the other hand, 3-dimensional counterexamples in any Kodaira dimension ≥ 0 were constructed by Benoist and Ottem [BO20] (these necessarily satisfy $H^0(X, \omega_X) = 0$ in Kodaira dimension 0).

It follows from weak factorization that the cokernel $Z^4(X)$ of the cycle class map is a birational invariant. Hence, the integral Hodge conjecture in codimension 2 is true for all rational varieties. This leads to the question whether there exist rationally connected or even unirational counterexamples to the integral Hodge conjecture in codimension 2. No such examples of dimension 3 can exist by [Voi06].

In dimension 4, Schreieder [Sch19] was able to construct a unirational variety violating the integral Hodge conjecture. His proof is based on an abstract description of $Z^4(X)$ in terms of unramified cohomology from [CTV12]. Concretely, we have

$$Z^4(X)[m] \cong \frac{H^3_{\mathrm{nr}}(X, \mathbb{Z}/m)}{H^3_{\mathrm{nr}}(X, \mathbb{Z}) \otimes \mathbb{Z}/m}$$

for all $m \geq 2$. If $\mathrm{CH}_0(X)$ is supported on a surface, we have $H^3_{\mathrm{nr}}(X, \mathbb{Z}) = 0$ and the above formula simplifies to $Z^4(X)[m] \cong H^3_{\mathrm{nr}}(X, \mathbb{Z}/m)$. In [Sch19], Schreieder constructed a singular conic bundle over $\mathbb{P}^3$ and was able to prove the existence of a non-zero class in $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2)$ for any smooth resolution X of this conic bundle. Therefore, the integral Hodge conjecture fails for any such X.

Since this argument uses the above description of $Z^4(X)[2]$, it is unclear whether the failure of the integral Hodge conjecture stems from a torsion cohomology class, as in the original counterexample by Atiyah and Hirzebruch, or from a non-torsion cohomology class, as in Kollár's examples discussed earlier. As shown in [Sch23], this question is equivalent to the problem whether the corresponding unramified class in $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2)$ can be represented by a global class in $H^3(X, \mathbb{Z}/2)$.

In chapter 5, we study Schreieder's example from a different, more concrete perspective. We carry out the first steps towards answering the open question raised in the preceding paragraph: First, we construct a smooth conic X birational to the given singular model. After that, we carefully study the corresponding unramified cohomology class on X and describe it expicitly via Borel–More homology. This allows to provide a geometric explanation for its unramifiedness. Furthermore, we reduce the question of representability by a global class to the cohomological vanishing of an explicitly given algebraic cycle. This algebraic cycle represents twice the non-algebraic Hodge class in Schreieder's example.

Our approach is unusual in the sense that most previous arguments involving unramified cohomology were quite abstract, whereas in chapter 5 we determine conrete equations for real submanifolds representing unramified cohomology classes. Along the way, we obtain a geometric description for the norm residue map over $\mathbb{C}$, which might be of independent interest.

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2. Constructing Hodge diamonds modulo m in characteristic zero

Abstract

For any integer $m \geq 2$ and any dimension $n \geq 1$, we show that any n-dimensional Hodge diamond with values in $\mathbb{Z}/m\mathbb{Z}$ is attained by the Hodge numbers of an n-dimensional smooth complex projective variety. As a corollary, there are no polynomial relations among the Hodge numbers of n-dimensional smooth complex projective varieties besides the ones induced by the Hodge symmetries, which answers a question raised by Kollár in 2012.

This chapter is based on [PS19], which was joint work with Stefan Schreieder.

2.1. Introduction

Hodge theory allows one to decompose the k-th Betti cohomology of an n-dimensional compact Kähler manifold X into its (p,q)-pieces for all $0 \leq k \leq 2n$:

$$H^k(X,\mathbb{C}) = \bigoplus_{\substack{p+q=k \\ 0\leq p,q\leq n}} H^{p,q}(X)\ , \quad \overline{H^{p,q}(X)} = H^{q,p}(X)\ .$$

The $\mathbb{C}$-linear subspaces $H^{p,q}(X)$ are naturally isomorphic to the Dolbeault cohomology groups $H^q(X,\Omega_X^p)$.

The integers $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ for $0 \leq p, q \leq n$ are called Hodge numbers. One usually arranges them in the so called Hodge diamond:

$$\begin{array}{ccccccccc}
 & & & & h^{n,n} & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & \iddots & & \vdots & & \ddots & & \\
 & h^{n,1} & & & & & & h^{1,n} & \\
h^{n,0} & & h^{n-1,1} & & \cdots & & h^{1,n-1} & & h^{0,n} \\
 & h^{n-1,0} & & & & & & h^{0,n-1} & \\
 & & \ddots & & \vdots & & \iddots & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & h^{0,0} & & & &
\end{array}$$

The sum of the k-th row of the Hodge diamond equals the k-th Betti number. We always assume that a Kähler manifold is compact and connected, so we have $h^{0,0} = h^{n,n} = 1$.

Complex conjugation and Serre duality induce the symmetries

$$h^{p,q} = h^{q,p} = h^{n-p,n-q} \quad \text{for all} \quad 0 \leq p, q \leq n \; . \tag{2.1}$$

Additionally, we have the Lefschetz inequalities

$$h^{p,q} \leq h^{p+1,q+1} \quad \text{for} \quad p + q < n \; . \tag{2.2}$$

While Hodge theory places severe restrictions on the geometry and topology of Kähler manifolds, Simpson points out in [Sim04] that very little is known to which extent the theoretically possible phenomena actually occur. This leads to the following construction problem for Hodge numbers:

Question 2.1. *Let $(h^{p,q})_{0 \leq p,q \leq n}$ be a collection of non-negative integers with $h^{0,0} = 1$ obeying the Hodge symmetries (2.1) and the Lefschetz inequalities (2.2). Does there exist a Kähler manifold X such that $h^{p,q}(X) = h^{p,q}$ for all $0 \leq p, q \leq n$?*

After results in dimensions two and three (see e. g. [Hun89]), significant progress has been made by Schreieder [Sch15]. For instance, it is shown in [Sch15, Theorem 3] that the above construction problem is fully solvable for large parts of the Hodge diamond in arbitrary dimensions. In particular, the Hodge numbers in a given weight k may be arbitrary (up to a quadratic lower bound on $h^{p,p}$ if $k = 2p$ is even) and so the outer Hodge numbers can be far larger than the inner Hodge numbers (see [Sch15, Theorem 1]), contradicting earlier expectations formulated in [Sim04]. Weaker results with simpler proofs, concerning the possible Hodge numbers in a given weight, have later been obtained by Arapura [Ara16].

In [Sch15], it was also observed that one cannot expect a positive answer to Question 2.1 in its entirety. For example, any 3-dimensional Kähler manifold X with $h^{1,1}(X) = 1$ and $h^{2,0}(X) \geq 1$ satisfies $h^{2,1}(X) < 12^6 \cdot h^{3,0}(X)$, see [Sch15, Proposition 28]. Therefore, a complete classification of all possible Hodge diamonds of Kähler manifolds or smooth complex projective varieties seems hopelessly complicated.

While these inequalities aggravate the construction problem for Hodge numbers, one might ask whether there also exist number theoretic obstructions for possible Hodge diamonds. For example, the Chern numbers of Kähler manifolds satisfy certain congruences due to integrality conditions implied by the Hirzebruch–Riemann–Roch theorem.

For an arbitrary integer $m \geq 2$, let us consider the Hodge numbers of a Kähler manifold in $\mathbb{Z}/m\mathbb{Z}$, which forces all inequalities to disappear. The purpose of this chapter is to show that Question 2.1 is modulo m completely solvable even for smooth complex projective varieties:

Theorem 2.2. *Let $m \geq 2$ be an integer. For any integer $n \geq 1$ and any collection of integers $(h^{p,q})_{0 \leq p,q \leq n}$ such that $h^{0,0} = 1$ and $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ for $0 \leq p, q \leq n$, there exists a smooth complex projective variety X of dimension n such that*

$$h^{p,q}(X) \equiv h^{p,q} \pmod{m}$$

for all $0 \leq p, q \leq n$.

Therefore, the Hodge numbers of Kähler manifolds do not follow any number theoretic rules, and the behaviour of smooth complex projective varieties is the same in this aspect.

As a consequence of Theorem 2.2, we show:

Corollary 2.3. *Up to the Hodge symmetries* (2.1)*, there are no polynomial relations among the Hodge numbers of smooth complex projective varieties of the same dimension.*

In particular, there are no polynomial relations in the strictly larger class of Kähler manifolds, which was a question raised by Kollár after a colloquium talk of Kotschick at the University of Utah in fall 2012. For linear relations among Hodge numbers, this question was settled in work of Kotschick and Schreieder [KS13].

We call the Hodge numbers $h^{p,q}(X)$ with $p \in \{0, n\}$ or $q \in \{0, n\}$ (i. e. the ones placed on the border of the Hodge diamond) the *outer Hodge numbers* of X and the remaining ones the *inner Hodge numbers*. Note that the outer Hodge numbers are birational invariants and are thus determined by the birational equivalence class of X.

Our proof shows (see Theorem 2.5 below) that any smooth complex projective variety is birational to a smooth complex projective variety with prescribed inner Hodge numbers in $\mathbb{Z}/m\mathbb{Z}$. As a corollary, there are no polynomial relations among the inner Hodge numbers within a given birational equivalence class. This is again a generalization of the corresponding result for linear relations obtained in [KS13, Theorem 2].

The proof of Theorem 2.2 can thus be divided into two steps: First we solve the construction problem modulo m for the outer Hodge numbers. This is done in Section 2.2. Then we show the aforementioned result that the inner Hodge numbers can be adjusted arbitrarily in $\mathbb{Z}/m\mathbb{Z}$ via birational equivalences (in fact, via repeated blow-ups). This is done in Section 2.3. Finally, in Section 2.4 we deduce that no non-trivial polynomial relations between Hodge numbers exist, thus answering Kollár's question.

2.2. Outer Hodge numbers

We prove the following statement via induction on the dimension $n \geq 1$.

Proposition 2.4. *For any collection of integers $(h^{p,0})_{1 \leq p \leq n}$, there exists a smooth complex projective variety X_n of dimension n together with a very ample line bundle L_n on X_n such that*

$$h^{p,0}(X_n) \equiv h^{p,0} \pmod{m}$$

for all $1 \leq p \leq n$ and

$$\chi(L_n^{-1}) \equiv 1 \pmod{m} .$$

Proof. We take X_1 to be a curve of genus g where $g \equiv h^{1,0} \pmod{m}$. Further, we take L_1 to be a line bundle of degree d on X_1 where $d > 2g$ and $d \equiv -g \pmod{m}$. Then L_1 is very ample and by the Riemann–Roch theorem we have $\chi(L_1^{-1}) \equiv 1 \pmod{m}$.

Now let $n > 1$. We define a collection of integers $(k^{p,0})_{-1 \leq p \leq n-1}$ recursively via

$$k^{-1,0} = 0 , \quad k^{0,0} = 1 , \quad k^{p,0} = h^{p,0} - 2k^{p-1,0} - k^{p-2,0} \text{ for } 1 \leq p \leq n-1 .$$

We choose X_{n-1} and L_{n-1} by induction hypothesis such that $h^{p,0}(X_{n-1}) \equiv k^{p,0} \pmod{m}$ for all $1 \le p \le n-1$.

Let E be a smooth elliptic curve and let L be a very ample line bundle of degree d on E such that $d \equiv 1 \pmod{m}$. Let e be a positive integer such that

$$e \equiv 1 + \sum_{p=1}^{n} (-1)^p h^{p,0} \pmod{m} .$$

Let $X_n \subset X_{n-1} \times E \times E$ be a hypersurface defined by a general section of the very ample line bundle

$$P_n = \mathrm{pr}_1^* L_{n-1} \otimes \mathrm{pr}_2^* L^{m-1} \otimes \mathrm{pr}_3^* L^e$$

on $X_{n-1} \times E \times E$. By Bertini's theorem, we may assume X_n to be smooth and irreducible. Let L_n be the restriction to X_n of the very ample line bundle

$$Q_n = \mathrm{pr}_1^* L_{n-1} \otimes \mathrm{pr}_2^* L \otimes \mathrm{pr}_3^* L$$

on $X_{n-1} \times E \times E$. Then L_n is again very ample.

By the Lefschetz hyperplane theorem, we have

$$h^{p,0}(X_n) = h^{p,0}(X_{n-1} \times E \times E)$$

for all $1 \le p \le n-1$. Since the Hodge diamond of $E \times E$ is

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} ,$$

Künneth's formula yields

$$\begin{aligned} h^{p,0}(X_n) &= h^{p,0}(X_{n-1}) + 2h^{p-1,0}(X_{n-1}) + h^{p-2,0}(X_{n-1}) \\ &\equiv k^{p,0} + 2k^{p-1,0} + k^{p-2,0} \pmod{m} \\ &= h^{p,0} \end{aligned}$$

for all $1 \le p \le n-1$. Therefore, it only remains to show that $h^{n,0}(X_n) \equiv h^{n,0} \pmod{m}$ and $\chi(L_n^{-1}) \equiv 1 \pmod{m}$. Since

$$\chi(\mathcal{O}_{X_n}) = 1 + \sum_{p=1}^{n} (-1)^p h^{p,0}(X_n) ,$$

the congruence $h^{n,0}(X_n) \equiv h^{n,0} \pmod{m}$ is equivalent to $\chi(\mathcal{O}_{X_n}) \equiv e \pmod{m}$.

By definition of X_n, the ideal sheaf on $X_{n-1} \times E \times E$ of regular functions vanishing on X_n is isomorphic to the sheaf of sections of the dual line bundle P_n^{-1}. Hence, there is a short exact sequence

$$0 \to P_n^{-1} \to \mathcal{O}_{X_{n-1}\times E\times E} \to i_*\mathcal{O}_{X_n} \to 0 \tag{2.3}$$

of sheaves on $X_{n-1} \times E \times E$ where $i\colon X_n \to X_{n-1} \times E \times E$ denotes the inclusion. Together with Künneth's formula and the Riemann–Roch theorem, we obtain

$$\begin{aligned}\chi(\mathcal{O}_{X_n}) &= \chi(\mathcal{O}_{X_{n-1}\times E\times E}) - \chi(P_n^{-1}) \\ &= \chi(\mathcal{O}_{X_{n-1}})\underbrace{\chi(\mathcal{O})^2}_{=0} - \underbrace{\chi(L_{n-1}^{-1})}_{\equiv 1}\underbrace{\chi(L^{1-m})}_{\equiv 1}\underbrace{\chi(L^{-e})}_{\equiv -e} \\ &\equiv e \pmod{m}\ .\end{aligned}$$

Tensoring (2.3) with Q_n^{-1} yields the short exact sequence

$$0 \to P_n^{-1} \otimes Q_n^{-1} \to Q_n^{-1} \to i_*i^*Q_n^{-1} \to 0$$

and thus

$$\begin{aligned}\chi(L_n^{-1}) &= \chi(Q_n^{-1}) - \chi(P_n^{-1} \otimes Q_n^{-1}) \\ &= \underbrace{\chi(L_{n-1}^{-1})}_{\equiv 1}\underbrace{\chi(L^{-1})^2}_{\equiv 1} - \chi(L_{n-1}^{-2})\underbrace{\chi(L^{-m})}_{\equiv 0}\chi(L^{-e-1}) \\ &\equiv 1 \pmod{m}\ .\end{aligned}$$

This finishes the induction step. □

2.3. Inner Hodge numbers

We now show the following result, which significantly improves [KS13, Theorem 2].

Theorem 2.5. *Let X be a smooth complex projective variety of dimension n and let $(h^{p,q})_{1\le p,q\le n-1}$ be any collection of integers such that $h^{p,q} = h^{q,p} = h^{n-p,n-q}$ for $1 \le p, q \le n-1$. Then X is birational to a smooth complex projective variety X' such that*

$$h^{p,q}(X') \equiv h^{p,q} \pmod{m}$$

for all $1 \le p, q \le n-1$.

Together with Proposition 2.4, this will complete the proof of Theorem 2.2.

Let us recall the following result on blow-ups, see e. g. [Voi03, Theorem 7.31]: If $\widetilde{X}$ denotes the blow-up of a Kähler manifold X along a closed submanifold $Z \subset X$ of codimension c, we have

$$H^{p,q}(\widetilde{X}) \cong H^{p,q}(X) \oplus \bigoplus_{i=1}^{c-1} H^{p-i,q-i}(Z) .$$

Therefore,

$$h^{p,q}(\widetilde{X}) = h^{p,q}(X) + \sum_{i=1}^{c-1} h^{p-i,q-i}(Z) . \tag{2.4}$$

In order to prove Theorem 2.5, we first show that we may assume that X contains certain subvarieties, without modifying its Hodge numbers modulo m.

Lemma 2.6. *Let X be a smooth complex projective variety of dimension n. Let $r, s \geq 0$ be integers such that $r + s \leq n - 1$. Then X is birational to a smooth complex projective variety X' of dimension n such that $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$ for all $0 \leq p, q \leq n$ and such that X' contains at least m disjoint smooth closed subvarieties that are all isomorphic to a projective bundle of rank r over $\mathbb{P}^s$.*

Proof. We first blow up X in a point and denote the result by $\widetilde{X}$. The exceptional divisor is a subvariety in $\widetilde{X}$ isomorphic to $\mathbb{P}^{n-1}$. In particular, $\widetilde{X}$ contains a copy of $\mathbb{P}^s \subset \mathbb{P}^{n-1}$. Now we blow up $\widetilde{X}$ along $\mathbb{P}^s$ to obtain $\widehat{X}$. The exceptional divisor in $\widehat{X}$ is the projectivization of the normal bundle of $\mathbb{P}^s$ in $\widetilde{X}$. Since $\mathbb{P}^s$ is contained in a smooth closed subvariety of dimension $r + s + 1$ in $\widetilde{X}$ (choose either $\mathbb{P}^{r+s+1} \subset \mathbb{P}^{n-1}$ if $r + s < n - 1$ or $\widetilde{X}$ if $r + s = n - 1$), the normal bundle of $\mathbb{P}^s$ in $\widetilde{X}$ contains a vector subbundle of rank $r + 1$, and hence its projectivization contains a projective subbundle of rank r. Therefore, $\widehat{X}$ admits a subvariety isomorphic to the total space of a projective bundle of rank r over $\mathbb{P}^s$.

By (2.4), the above construction only has an additive effect on the Hodge diamond, i. e. the differences between respective Hodge numbers of $\widehat{X}$ and X are constants independent of X. Hence, we may apply the above construction $m - 1$ more times to obtain a smooth complex projective variety X' containing m disjoint copies of the desired projective bundle and satisfying $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$. ☐

In the following, we consider the primitive Hodge numbers

$$l^{p,q}(X) = h^{p,q}(X) - h^{p-1,q-1}(X)$$

for $p+q \leq n$. Clearly, it suffices to show Theorem 2.5 for a given collection $(l^{p,q})_{(p,q)\in I}$ of primitive Hodge numbers instead, where

$$I = \{(p,q) \mid 1 \leq p \leq q \leq n-1 \text{ and } p+q \leq n\} .$$

This is because one can get back the original Hodge numbers from the primitive Hodge numbers via the relation

$$h^{p,q}(X) = h^{0,q-p}(X) + \sum_{i=1}^{p} l^{i,q-p+i}(X)$$

for $p \leq q$ and $p+q \leq n$, and $h^{0,q-p}(X)$ is a birational invariant.

We define a total order $\prec$ on I via

$$(r,s) \prec (p,q) \iff r+s < p+q \text{ or } (r+s = p+q \text{ and } s < q) .$$

Proposition 2.7. *Let X be a smooth complex projective variety of dimension n. Let $(r,s) \in I$. Then X is birational to a smooth complex projective variety X' of dimension n such that*

$$l^{r,s}(X') \equiv l^{r,s}(X) + 1 \pmod m$$

and

$$l^{p,q}(X') \equiv l^{p,q}(X) \pmod m$$

for all $(p,q) \in I$ with $(r,s) \prec (p,q)$.

Proof. By Lemma 2.6, we may assume that X contains m disjoint copies of a projective bundle of rank $r-1$ over $\mathbb{P}^{s-r+1}$. Therefore, it is possible to blow up X along a projective bundle B_d of rank $r-1$ over smooth hypersurfaces $Y_d \subset \mathbb{P}^{s-r+1}$ of degree d (in case of $r = s$, Y_d just consists of d distinct points in $\mathbb{P}^1$) and we may repeat this procedure m times and with different values for d. The Hodge numbers of B_d are the same as for the trivial bundle $Y_d \times \mathbb{P}^{r-1}$, see e. g. [Voi03, Lemma 7.32].

By the Lefschetz hyperplane theorem, the Hodge diamond of Y_d is the sum of the Hodge diamond of $Y_1 \cong \mathbb{P}^{s-r}$, having non-zero entries only in the middle column, and of a Hodge diamond depending on d, having non-zero entries only in the middle row. It is well known (e. g. by computing Euler characteristics as in Section 2.2) that the two outer entries of this middle row are precisely $\binom{d-1}{s-r+1}$.

Now we blow up X once along B_{s-r+2} and $m-1$ times along B_1 and denote the resulting smooth complex projective variety by X'. Due to (2.4) and Künneth's formula, this construction affects the Hodge numbers modulo m in the same way as if we would blow up a single subvariety $Z \times \mathbb{P}^{r-1} \subset X$, where Z is a (formal)

$(s-r)$-dimensional Kähler manifold whose Hodge diamond is concentrated in the middle row and has outer entries equal to $\binom{s-r+2-1}{s-r+1} = 1$. In particular, we have $h^{p,q}(Z \times \mathbb{P}^{r-1}) = 0$ unless $s - r \leq p + q \leq s + r - 2$ (and $p + q$ has the same parity as $s - r$) and $|p - q| \leq s - r$. On the other hand, $h^{p,q}(Z \times \mathbb{P}^{r-1}) = 1$ if $s - r \leq p + q \leq s + r - 2$ and $|p - q| = s - r$.

Taking differences in (2.4), it follows that

$$l^{p,q}(X') \equiv l^{p,q}(X) + h^{p-1,q-1}(Z \times \mathbb{P}^{r-1}) - h^{p-n+s-1,q-n+s-1}(Z \times \mathbb{P}^{r-1}) \pmod m$$

for all $p + q \leq n$. But we have

$$(p - n + s - 1) + (q - n + s - 1) = p + q - 2n + 2s - 2 \leq 2s - n - 2 \leq s - r - 2$$

and hence $h^{p-n+s-1,q-n+s-1}(Z \times \mathbb{P}^{r-1}) = 0$ for all $(p, q) \in I$ by the above remark.

Further,

$$l^{r,s}(X') \equiv l^{r,s}(X) + h^{r-1,s-1}(Z \times \mathbb{P}^{r-1}) = l^{r,s}(X) + 1 \pmod m$$

since $s - r \leq (r - 1) + (s - 1) \leq s + r - 2$ and $|r - s| = s - r$.

Finally, $r + s < p + q$ implies $(p - 1) + (q - 1) > s + r - 2$, while $r + s = p + q$ and $s < q$ imply $|p - q| > s - r$, so we have $h^{p-1,q-1}(Z \times \mathbb{P}^{r-1}) = 0$ in both cases and thus

$$l^{p,q}(X') \equiv l^{p,q}(X) + h^{p-1,q-1}(Z \times \mathbb{P}^{r-1}) = l^{p,q}(X) \pmod m$$

for all $(p, q) \in I$ with $(r, s) \prec (p, q)$. □

Proof of Theorem 2.5. The statement is an immediate consequence of applying Proposition 2.7 inductively $t_{p,q}$ times to each $(p, q) \in I$ in the descending order induced by $\prec$, where $t_{p,q} \equiv l^{p,q} - l^{p,q}(X_{p,q}) \pmod m$ and $X_{p,q}$ is the variety obtained in the previous step. □

2.4. Polynomial relations

The following principle seems to be classical.

Lemma 2.8. *Let $N \geq 1$ and let $S \subset \mathbb{Z}^N$ be a subset such that its reduction modulo m is the whole of $(\mathbb{Z}/m\mathbb{Z})^N$ for infinitely many integers $m \geq 2$. If $f \in \mathbb{C}[x_1, \dots, x_N]$ is a polynomial vanishing on S, then $f = 0$.*

Proof. Let $f \in \mathbb{C}[x_1, \ldots, x_N]$ be a non-zero polynomial vanishing on S. By choosing a $\mathbb{Q}$-basis of $\mathbb{C}$ and a $\mathbb{Q}$-linear projection $\mathbb{C} \to \mathbb{Q}$ which sends a non-zero coefficient of f to 1, we see that we may assume that the coefficients of f are rational, hence even integral. Since $f \neq 0$, there exists a point $z \in \mathbb{Z}^N$ such that $f(z) \neq 0$. Choose an integer $m \geq 2$ from the assumption which does not divide $f(z)$. Then $f(z) \not\equiv 0 \pmod{m}$. However, we have $z \equiv s \pmod{m}$ for some $s \in S$ and thus $f(z) \equiv f(s) = 0 \pmod{m}$, because $f \in \mathbb{Z}[x_1, \ldots, x_N]$. This is a contradiction. □

Proof of Corollary 2.3. This follows by applying Lemma 2.8 to the set S of possible Hodge diamonds, where we consider only a non-redundant quarter of the diamond to take the Hodge symmetries into account. Theorem 2.2 guarantees that the reductions of S modulo m are surjective even for all integers $m \geq 2$. □

In the same way Theorem 2.2 implies Corollary 2.3, Theorem 2.5 yields the following.

Corollary 2.9. *There are no non-trivial polynomial relations among the inner Hodge numbers of all smooth complex projective varieties in any given birational equivalence class.*

3. Constructing Hodge diamonds modulo m in positive characteristic

Abstract

Let k be an algebraically closed field of positive characteristic. For any integer $m \geq 2$, we show that the Hodge numbers of a smooth projective k-variety can take on any combination of values modulo m, subject only to Serre duality. In particular, there are no non-trivial polynomial relations between the Hodge numbers.

This chapter is based on [vDdBP20], which was joint work with Remy van Dobben de Bruyn.

3.1. Introduction

The Hodge numbers $h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega_X^p)$ of an n-dimensional smooth projective variety X over $\mathbb{C}$ satisfy the following conditions:

(1) $h^{0,0}(X) = 1$ (connectedness);
(2) $h^{p,q}(X) = h^{n-p,n-q}(X)$ for all $0 \leq p, q \leq n$ (Serre duality);
(3) $h^{p,q}(X) = h^{q,p}(X)$ for all $0 \leq p, q \leq n$ (Hodge symmetry).

Kotschick and Schreieder showed [KS13, Thm. 1, consequence (2)] that the only linear relations among the Hodge numbers that are satisfied by *all* smooth projective $\mathbb{C}$-varieties of dimension n are the ones induced by (1), (2), and (3).

In positive characteristic, Hodge symmetry (3) does not always hold [Ser58, Prop. 16], but Serre duality (2) is still true. Van Dobben de Bruyn proved that (1) and (2) are indeed the only universal *linear* relations among the Hodge numbers of n-dimensional smooth projective k-varieties if $\operatorname{char} k > 0$ [vDdB21, Thm. 1].

In [PS19, Thm. 2], the author and Schreieder solved the construction problem over $\mathbb{C}$ for Hodge diamonds modulo an arbitrary integer $m \geq 2$. This means that for any dimension n and any collection of integers satisfying the conditions (1), (2), and (3), there exists a smooth projective $\mathbb{C}$-variety of dimension n whose Hodge numbers agree with the given integers modulo m. As a corollary, there are no non-trivial *polynomial* relations among the Hodge numbers, which strengthens the result from [KS13] on linear relations.

In this chapter, we solve the construction problem for Hodge diamonds modulo m in positive characteristic:

Theorem 3.1. *Let k be an algebraically closed field of positive characteristic, and let $m \geq 2$ and $n \geq 0$ be integers. Let $(a^{p,q})_{0 \leq p,q \leq n}$ be any collection of integers such that $a^{0,0} = 1$ and $a^{p,q} = a^{n-p,n-q}$ for all $0 \leq p, q \leq n$. Then there exists a smooth projective k-variety X of dimension n such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all $0 \leq p, q \leq n$.

In analogy to [PS19, Cor. 3], it follows that there are no polynomial relations among the Hodge numbers in positive characteristic besides (1) and (2) (see Corollary 3.18). This extends the result from [vDdB21, Thm. 1] on linear relations.

Theorem 3.1 also shows that Hodge symmetry may fail arbitrarily badly in positive characteristic. For any dimension n and all $0 \leq p < q \leq n$ with $p + q \neq n$, the Hodge numbers $h^{p,q}$ and $h^{q,p}$ can not only be different, but can even be incongruent modulo any integer $m \geq 2$. Note that Hodge symmetry (3) is a consequence of Serre duality (2) if $p + q = n$, and thus always holds in the middle row of the Hodge diamond.

A complete classification of the possible Hodge diamonds of smooth projective k-varieties, i. e. a version of Theorem 3.1 without the "modulo m" part, seems to be very hard already when Hodge symmetry is true; see [Sch15] for strong partial results on this in characteristic zero.

The structure of our proof is similar to [PS19], with some improvements. First we solve the construction problem modulo m for the outer Hodge numbers, i. e. the Hodge numbers $h^{p,q}$ with $p \in \{0, n\}$ or $q \in \{0, n\}$ (see Proposition 3.10). Then we prove that for any smooth projective k-variety, there exists a sequence of blowups in smooth centres such that the inner Hodge numbers of the blowup, i. e. the Hodge numbers $h^{p,q}$ with $1 \leq p, q \leq n-1$, attain any given values in $\mathbb{Z}/m$ satisfying Serre duality (2). Hence we obtain the following result, which might be of independent interest:

Theorem 3.2. *Let k be an algebraically closed field of positive characteristic, and let $m \geq 2$ and $n \geq 0$ be integers. Let X be a smooth projective k-variety of dimension n and let $(a^{p,q})_{1 \leq p,q \leq n-1}$ be any collection of integers such that $a^{p,q} = a^{n-p,n-q}$ for all $1 \leq p, q \leq n-1$. Then there exists a smooth projective k-variety $\tilde{X}$ birational to X such that*

$$h^{p,q}(\tilde{X}) \equiv a^{p,q} \pmod{m}$$

for all $1 \leq p, q \leq n-1$.

The analogous result in characteristic zero was obtained in [PS19, Thm. 5]. The fact that all outer Hodge numbers are birational invariants in positive characteristic was proven by Chatzistamatiou and Rülling [CR11, Thm. 1], so Theorem 3.2 is the best possible statement. Again, it follows that the result from [vDdB21, Thm. 3] on linear birational invariants extends to polynomials (see Corollary 3.19).

In analogy with [vDdB21, Thm. 2], our constructions only need Serre's counterexample [Ser58, Prop. 16] to generate all Hodge asymmetry. While the structure of our argument is similar to [PS19], the absence of condition (3) in positive characteristic raises new difficulties for both the inner and the outer Hodge numbers. There is a quick proof of Theorem 3.2 assuming embedded resolution of singularities in positive characteristic, see Remark 3.16. The proof we present is similar, but does a little more work to avoid using embedded resolution. It relies on Maruyama's theory of elementary transformations of vector bundles.

In section 3.2, we state and prove some lemmas on Hodge numbers that are used later. The constructions for outer and inner Hodge numbers are carried out in section 3.3 and section 3.4, respectively. Finally, we deduce corollaries on polynomial relations in section 3.5.

Throughout this chapter, we fix an algebraically closed field k of positive characteristic and an integer $m \geq 2$.

3.2. Some lemmas on Hodge numbers

In this section, we collect some standard results on Hodge numbers that we will use repeatedly in the arguments. The only difference between the situation in characteristic zero [KS13, PS19] and positive characteristic [vDdB21] comes from asymmetry of Hodge diamonds, and as in [vDdB21] the only example we need is Serre's surface:

Theorem 3.3. *There exists a smooth projective k-variety S of dimension two such that $h^{1,0}(S) = 0$ and $h^{0,1}(S) = 1$.*

Proof. See [Ser58, Prop. 16], or [vDdB21, Prop. 1.4] for a short modern account. □

We use the following well-known formula for Hodge numbers under blowups. In characteristic zero, this corresponds to equation (2.4), see [Voi03, Theorem 7.31].

Lemma 3.4. *Let X be a smooth projective k-variety, let $Z \subseteq X$ be a smooth subvariety of codimension r, and let $\tilde{X} \to X$ be the blowup of X at Z. Then the Hodge numbers of $\tilde{X}$ satisfy*

$$h^{p,q}(\tilde{X}) = h^{p,q}(X) + \sum_{i=1}^{r-1} h^{p-i,q-i}(Z).$$

A consequence that will be used repeatedly is that any blowup construction carried out m times does not change the Hodge numbers modulo m.

Proof of Lemma 3.4. See for example [Gro85, Cor. IV.1.1.11]. As noted by Achinger and Zdanowicz [AZ17, Cor. 2.8], it is also an immediate consequence of Voevodsky's motivic blowup formula [Voe00, Prop. 3.5.3] and Chatzistamatiou–Rülling's action of Chow groups on Hodge cohomology [CR11]. □

The Hodge numbers of a product $X_1 \times X_2$ can be easily described in terms of the Hodge numbers of X_1 and X_2 by a Künneth-type formula.

Lemma 3.5. *Let X_1 and X_2 be smooth projective k-varieties. Then the Hodge numbers of $X := X_1 \times X_2$ are given by*

$$h^{p,q}(X) = \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} h^{p_1,q_1}(X_1) \cdot h^{p_2,q_2}(X_2).$$

Proof. We have $\Omega_X = \pi_1^*\Omega_{X_1} \oplus \pi_2^*\Omega_{X_2}$ and thus

$$\Omega_X^p = \bigoplus_{p_1+p_2=p} \pi_1^*\Omega_{X_1}^{p_1} \otimes \pi_2^*\Omega_{X_2}^{p_2}.$$

Hence, using the classical Künneth formula for quasi-coherent sheaves we get

$$\begin{aligned} H^q(X, \Omega_X^p) &= \bigoplus_{p_1+p_2=p} H^q\left(X, \pi_1^*\Omega_{X_1}^{p_1} \otimes \pi_2^*\Omega_{X_2}^{p_2}\right) \\ &= \bigoplus_{\substack{p_1+p_2=p \\ q_1+q_2=q}} H^{q_1}\left(X_1, \Omega_{X_1}^{p_1}\right) \otimes H^{q_2}\left(X_2, \Omega_{X_2}^{p_2}\right). \end{aligned} \qquad \square$$

The next lemma provides a weak Lefschetz theorem for sufficiently ample hypersurfaces.

Lemma 3.6. *Let X be a smooth projective k-variety of dimension $n+1$ with a very ample line bundle $\mathscr{L} = \mathcal{O}_X(H)$. Let $d_0 \in \mathbb{Z}_{>0}$ such that $H^q(X, \Omega_X^p(-dH)) = 0$ when $d \geq d_0$ and $p+q \leq n$. Then any smooth divisor $Y \in |dH|$ with $d \geq d_0$ satisfies $h^{p,q}(Y) = h^{p,q}(X)$ when $p+q \leq n-1$.*

Proof. The short exact sequence

$$0 \to \Omega_X^p(-dH) \to \Omega_X^p \to \Omega_X^p|_Y \to 0$$

shows that for all $p+q \leq n-1$ and all $e \geq 0$, we have

$$H^q\Big(X, \Omega_X^p(-eH)\Big) = H^q\Big(Y, \Omega_X^p(-eH)|_Y\Big). \tag{3.1}$$

We will prove by induction on p that $H^q(Y, \Omega_X^p(-eH)|_Y) = H^q(Y, \Omega_Y^p(-eH))$ for all $e \geq 0$ and $p+q \leq n-1$. Together with (3.1) this proves the result by taking $e=0$. The base case $p=0$ is trivial since $\mathcal{O}_X|_Y = \mathcal{O}_Y$. For $p>0$, the inductive hypothesis, (3.1), and the assumption on d_0 imply

$$H^q\Big(Y, \Omega_Y^i(-eH)\Big) = H^q\Big(Y, \Omega_X^i(-eH)|_Y\Big) = H^q\Big(X, \Omega_X^i(-eH)\Big) = 0 \tag{3.2}$$

for $i+q \leq n-1$, $e \geq d_0$, and $i<p$. The conormal sequence

$$0 \to \mathcal{O}_Y(-Y) \to \Omega_X^1|_Y \to \Omega_Y^1 \to 0$$

gives a short exact sequence

$$0 \to \Omega_Y^{p-1}(-Y) \to \Omega_X^p|_Y \to \Omega_Y^p \to 0 \tag{3.3}$$

since $\mathcal{O}_Y(-Y)$ is a line bundle. Now (3.2) gives

$$H^q\Big(Y, \Omega_Y^{p-1}(-Y-eH)\Big) = H^q\Big(Y, \Omega_Y^{p-1}\left(-(d+e)H\right)\Big) = 0$$

for $p+q \leq n$ and $e \geq 0$. Thus, (3.3) shows that the natural map

$$H^q\Big(Y, \Omega_X^p(-eH)|_Y\Big) \to H^q\Big(Y, \Omega_Y^p(-eH)\Big)$$

is an isomorphism for $p+q \leq n-1$ and $e \geq 0$, as claimed. $\square$

Corollary 3.7. *Let X be a smooth projective k-variety of dimension $n+1$ with a very ample line bundle $\mathscr{L} = \mathcal{O}_X(H)$. Then any smooth divisor $Y \in |dH|$ with $d \gg 0$ satisfies $h^{p,q}(Y) = h^{p,q}(X)$ when $p+q \leq n-1$.*

Proof. By Serre vanishing, there exists $d_0 \in \mathbb{Z}$ such that $H^q(X, \Omega_X^p(-dH)) = 0$ for all $d \geq d_0$ and $q \leq n$. Then Lemma 3.6 gives the result. $\square$

Remark 3.8. If $\operatorname{char} k = 0$, then by Nakano vanishing we may take $d_0 = 1$ in Lemma 3.6. This recovers the usual proof of weak Lefschetz from Nakano vanishing. Similarly, if $\operatorname{char} k > 0$ and Nakano vanishing holds for X, then we may take $d_0 = 1$, but in general already Kodaira vanishing may fail in positive characteristic [Ray78].

For our application, it's useful to have some control over the Euler characteristic of $\mathscr{L}^{-1}$.

Lemma 3.9. *Let X be a smooth projective k-variety of dimension $n+1$ and let $e \in \mathbb{Z}$. Then, up to modifying X by blowups in smooth centres that do not change its Hodge numbers modulo m, we may assume that X admits a very ample line bundle $\mathscr{L} = \mathcal{O}_X(H)$ such that $\chi(X, \mathscr{L}^{-1}) \equiv e \pmod{m}$ and such that any smooth divisor $Y \in |H|$ satisfies $h^{p,q}(Y) = h^{p,q}(X)$ when $p+q \leq n-1$.*

Proof. Let $\pi\colon \tilde{X} \to X$ be a blowup in m distinct points $p_1, \ldots, p_m \in X$. Then the blowup formula for Hodge numbers (Lemma 3.4) gives $h^{p,q}(\tilde{X}) \equiv h^{p,q}(X) \pmod{m}$. Let $E_i = \pi^{-1}(p_i)$ be the exceptional divisors, and for $r \in \{0, \ldots, m\}$ write $E_{\leq r} = E_1 + \ldots + E_r$. Then the short exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-E_{\leq r}) \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{E_{\leq r}} \to 0$$

shows that

$$\chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E_{\leq r})) = \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}) - \sum_{i=1}^{r} \chi(E_i, \mathcal{O}_{E_i}) = \chi(X, \mathcal{O}_X) - r.$$

Take $r \in \{0, \ldots, m-1\}$ with $r \equiv \chi(X, \mathcal{O}_X) - e \pmod{m}$.

Let $\mathscr{M}$ be an ample line bundle on $\tilde{X}$. By Serre vanishing there exists $a_0 \in \mathbb{Z}$ such that for all $a \geq a_0$, the line bundle $\mathscr{L} = \mathscr{M}^{\otimes a} \otimes \mathcal{O}_{\tilde{X}}(E_{\leq r})$ is very ample and satisfies

$$H^q(X, \Omega_X^p \otimes \mathscr{L}^{-d}) = 0 \tag{3.4}$$

for $d > 0$ and $q \leq n$. Taking a divisible by the product of m and the denominators of the coefficients of the Hilbert polynomial $P(t) = \chi(\tilde{X}, \mathscr{M}^{\otimes t} \otimes \mathcal{O}_{\tilde{X}}(-E_{\leq r}))$, we see that

$$\chi(\tilde{X}, \mathscr{L}^{-1}) \equiv \chi(\tilde{X}, \mathcal{O}_{\tilde{X}}(-E_{\leq r})) \equiv e \pmod{m}.$$

Finally, $\mathscr{L}$ satisfies weak Lefschetz by (3.4) and Lemma 3.6. $\square$

3.3. Outer Hodge numbers

In this section, we solve the construction problem for the outer Hodge numbers. Because of Serre duality and the fact that $h^{0,0} = 1$, it suffices to consider the Hodge numbers $h^{p,q}$ with $(p, q) \in J_n$, where

$$J_n = \{(1, 0), \dots, (n, 0), (0, 1), \dots, (0, n)\}.$$

The main result of this section is the following:

Proposition 3.10. *Let $n \geq 0$. For any given integers $a^{1,0}, \dots, a^{n,0}$ and $a^{0,1}, \dots, a^{0,n}$ with $a^{n,0} = a^{0,n}$, there exists a smooth projective k-variety X of dimension n such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all $(p, q) \in J_n$.

The construction will be carried out by induction on the dimension, using the weak Lefschetz results from Corollary 3.7 and Lemma 3.9.

Lemma 3.11. *Let $n, d \geq 0$ be integers such that $d \geq n-1$. If Proposition 3.10 holds in dimension d for $a^{1,0}, \dots, a^{d,0}$ and $a^{0,1}, \dots, a^{0,d}$ with $a^{d,0} = a^{0,d}$, then it also holds in dimension n for $a^{1,0}, \dots, a^{n-1,0}, b$ and $a^{0,1}, \dots, a^{0,n-1}, b$ for any $b \in \mathbb{Z}$.*

Proof. Let X be a smooth projective k-variety of dimension d with the given Hodge numbers $a^{p,q}$. We may assume that $d \geq n + 1$ by multiplying X with $\mathbb{P}^2$, which does not change its outer Hodge numbers in degree $\leq n - 1$. By repeatedly replacing X by a smooth hyperplane section of sufficiently high degree, we may further assume that $d = n + 1$ by Corollary 3.7. By Lemma 3.9, after possibly replacing X by a blowup that does not change its Hodge numbers modulo m, there exists a very ample line bundle $\mathscr{L}$ on X such that

$$\chi(X, \mathscr{L}^{-1}) \equiv (-1)^n(a^{0,n} - a^{0,n+1} - b) \pmod{m} \tag{3.5}$$

and such that a smooth section Y of $\mathscr{L}$ satisfies $h^{p,q}(Y) \equiv a^{p,q} \pmod{m}$ for $p+q \leq n-1$. The short exact sequence

$$0 \to \mathscr{L}^{-1} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

gives $\chi(X, \mathscr{L}^{-1}) = \chi(X, \mathcal{O}_X) - \chi(Y, \mathcal{O}_Y)$. Since $h^{0,q}(X) = h^{0,q}(Y)$ for $q \leq n-1$, we conclude that

$$\begin{aligned}\chi(X, \mathscr{L}^{-1}) &= (-1)^n h^{0,n}(X) + (-1)^{n+1} h^{0,n+1}(X) - (-1)^n h^{0,n}(Y)\\ &\equiv (-1)^n \left(a^{0,n} - a^{0,n+1} - h^{0,n}(Y)\right). \pmod{m}\end{aligned}$$

With (3.5) we get $h^{0,n}(Y) \equiv b \pmod{m}$, so Serre duality gives $h^{n,0}(Y) \equiv b \pmod{m}$. $\square$

Note that in characteristic zero, Lemma 3.11 immediately implies Proposition 3.10, giving an alternative approach to a variant of [PS19, Prop. 4]. In positive characteristic, however, the failure of Hodge symmetry raises new difficulties, since e. g. $h^{n-1,0} = h^{0,n-1}$ is true for varieties of dimension $n-1$ but not for all varieties of dimension n. This problem is solved in the following construction, which together with Lemma 3.11 implies Proposition 3.10.

Lemma 3.12. *Let $n \geq 2$. For any given integers $a^{0,1}, \ldots, a^{0,n-1}$ and $a^{1,0}, \ldots, a^{n-1,0}$, there exists a smooth projective k-variety X of dimension $\geq n-1$ such that*

$$h^{p,q}(X) \equiv a^{p,q} \pmod{m}$$

for all $(p,q) \in J_{n-1}$.

Note that we do not assume $a^{0,n-1} = a^{n-1,0}$ here, so we typically need $\dim X \geq n$.

Proof of Lemma 3.12. First consider the case $n = 2$. Let E be an elliptic curve and let S be the surface from Theorem 3.3. Choose $i \geq 0$ and $j \geq 1$ with $i \equiv a^{0,1} - a^{1,0} \pmod{m}$ and $j \equiv a^{1,0} \pmod{m}$, and set $X = S^i \times E^j$. Then it follows from Künneth's formula (Lemma 3.5) that $h^{0,1}(X) \equiv i + j \equiv a^{0,1} \pmod{m}$ and $h^{1,0}(X) \equiv j \equiv a^{1,0} \pmod{m}$.

Now assume $n \geq 3$. By Lemma 3.11, we may assume inductively that Proposition 3.10 holds in dimensions $\leq n-1$. Therefore, there exists a smooth projective variety Y of dimension $n-1$ with outer Hodge numbers

$$h^{p,q}(Y) \equiv \begin{cases} (-1)^q, & p = 0,\ 0 \leq q < n-1, \\ 0, & p = 0,\ q = n-1, \\ 0, & p > 0,\ q = 0. \end{cases} \pmod{m}.$$

By Proposition 3.10 in dimension 2, there exists a smooth projective surface S with outer Hodge numbers $h^{1,0}(S) \equiv h^{2,0}(S) \equiv h^{0,2}(S) \equiv 0 \pmod{m}$ and $h^{0,1}(S) \equiv 1 \pmod{m}$. The Künneth formula from Lemma 3.5 shows that $S \times Y$ has outer Hodge numbers $h^{p,q}(S \times Y) \equiv 0 \pmod{m}$ for $(p,q) \in J_{n-1}$, except $h^{0,0}(S \times Y) = 1$ and $h^{0,n-1}(S \times Y) \equiv (-1)^n \pmod{m}$.

Finally, by Proposition 3.10 in dimension $n-1$, there exists a smooth projective variety Z with outer Hodge numbers given by

$$h^{p,q}(Z) \equiv \begin{cases} a^{p,q}, & (p,q) \in J_{n-1} \setminus \{(0,n-1)\}, \\ a^{n-1,0}, & (p,q) = (0,n-1). \end{cases} \pmod{m}$$

Taking $X = Z \times (S \times Y)^i$ for $i \geq 0$ gives outer Hodge numbers

$$h^{p,q}(X) \equiv \begin{cases} a^{p,q}, & (p,q) \in J_{n-1} \setminus \{(0,n-1)\}, \\ a^{n-1,0} + (-1)^n i, & (p,q) = (0,n-1). \end{cases} \pmod m$$

The result follows by taking $i \equiv (-1)^n(a^{0,n-1} - a^{n-1,0}) \pmod m$. □

3.4. Inner Hodge numbers

The aim of this section is to prove Theorem 3.2, i. e. to modify the inner Hodge numbers of a smooth projective k-variety via successive blowups. We first show how to produce certain subvarieties with asymmetric Hodge numbers that we will blow up later.

Lemma 3.13. *Let X be a smooth projective k-variety of dimension n, let $b, c \in \mathbb{Z}$, and let $d \in \{2, \dots, n-2\}$. Then there exists a smooth projective variety $\tilde{X}$ and a birational morphism $\tilde{X} \to X$ obtained as a composition of blowups in smooth centres that does not change the Hodge numbers modulo m such that $\tilde{X}$ contains a smooth subvariety W of dimension d satisfying*

$$h^{d,0}(W) = h^{0,d}(W) \equiv 0 \pmod m \tag{3.6}$$

and

$$h^{d-1,0}(W) \equiv b, \quad h^{0,d-1}(W) \equiv c \pmod m. \tag{3.7}$$

Proof. Let $X_1 \to X$ be the blowup of X in a point. The assumption on d implies $n \geq 4$, so the exceptional divisor of X_1 contains $\mathbb{P}^3$. By Proposition 3.10, there exists a smooth projective surface S_0 such that $h^{2,0}(S_0) = h^{0,2}(S_0) \equiv 0 \pmod m$ and

$$h^{1,0}(S_0) \equiv b, \quad h^{0,1}(S_0) \equiv c \pmod m.$$

Choose a possibly singular surface $S_1 \subseteq \mathbb{P}^3$ birational to S_0. By embedded resolution of surfaces [Abh66, Thm. 9.1.3] (see also [Cut09, Thm. 1.2]), there exists a birational morphism $X_2 \to X_1$ obtained as a composition of blowups in smooth centres contained in $\mathbb{P}^3$ such that the strict transform S of S_1 is smooth. Since S is also birational to S_0, we have $h^{2,0}(S) = h^{0,2}(S) \equiv 0 \pmod m$ and

$$h^{1,0}(S) \equiv b, \quad h^{0,1}(S) \equiv c \pmod m.$$

Now consider the blowup $X_3 \to X_2$ in S. The exceptional divisor is a $\mathbb{P}^{n-3}$-bundle $\mathbb{P}_S(\mathscr{E})$ over S. Let $Z \subseteq \mathbb{P}^{n-3}$ be a smooth hypersurface of degree d in a linear subspace $\mathbb{P}^{d-1} \subseteq \mathbb{P}^{n-3}$; in particular, Z satisfies $h^{d-2,0}(Z) = h^{0,d-2}(Z) = 1$.

By Maruyama's theory of elementary transformations (see [vDdBP20, Proposition A.8]), there exists a diagram

$$\begin{tikzcd} & \widetilde{P} \arrow[dl, "f"'] \arrow[dr, "f'"] & & \\ S \times \mathbb{P}^{n-3} & & \mathbb{P}_S(\mathscr{E}) \arrow[r, hook] & X_3 \,, \end{tikzcd}$$

where f and f' are blowups in smooth centres Y and Y' respectively, such that $Y \cap (S \times Z)$ is smooth. Then the blowup $X_4 \to X_3$ in Y' contains the strict transform

$$W = \widetilde{S \times Z} = \mathrm{Bl}_{Y\cap(S\times Z)}(S \times Z)$$

of $S \times Z$ under f. Birational invariance of outer Hodge numbers (in the case of a blowup this is Lemma 3.4) and the Künneth formula (Lemma 3.5) give

$$\begin{aligned} h^{d,0}(W) = h^{0,d}(W) = h^{d,0}(S \times Z) = h^{2,0}(S)h^{d-2,0}(Z) &\equiv 0 \pmod m, \\ h^{d-1,0}(W) = h^{d-1,0}(S \times Z) = h^{2,0}(S)h^{d-3,0}(Z) + h^{1,0}(S)h^{d-2,0}(Z) &\equiv b \pmod m, \\ h^{0,d-1}(W) = h^{0,d-1}(S \times Z) = h^{0,2}(S)h^{0,d-3}(Z) + h^{0,1}(S)h^{0,d-2}(Z) &\equiv c \pmod m. \end{aligned}$$

Blowing up $m-1$ more points coming from X and repeating the above construction $m-1$ more times in each exceptional $\mathbb{P}^{n-1}$ separately, the blowup formula of Lemma 3.4 shows that the Hodge numbers of X do not change modulo m. □

Corollary 3.14. *Let X be a smooth projective k-variety of dimension n, let $b, c \in \mathbb{Z}$, and let $r \in \{1, \ldots, n-1\}$. Assume that $b = c$ if $r = 1$ or $r = n-1$. Then there exists a birational morphism $\tilde{X} \to X$ obtained by a sequence of blowups in smooth centres such that*

$$h^{r,1}(\tilde{X}) \equiv h^{r,1}(X) + b, \quad h^{1,r}(\tilde{X}) \equiv h^{1,r}(X) + c \pmod m$$

and

$$h^{p,1}(\tilde{X}) \equiv h^{p,1}(X), \quad h^{1,p}(\tilde{X}) \equiv h^{1,p}(X) \pmod m$$

for all $p > r$.

Proof. If $r \in \{2, \ldots, n-2\}$, then Lemma 3.13 shows that there exists a successive blowup $X' \to X$ that does not change the Hodge numbers modulo m such that X' contains a subvariety W of dimension r satisfying (3.6) and (3.7). Letting $\tilde{X} \to X'$ be the blowup in W gives the result by Lemma 3.4.

For $r = 1$, we consider the blowup in $i \geq 0$ points where $i \equiv b = c \pmod m$. Then the statement follows again from Lemma 3.4.

For $r = n-1$, we first blow up X in $i \geq 0$ points where $i \equiv b = c \pmod m$. Then, in each exceptional $\mathbb{P}^{n-1}$ we blow up a smooth hypersurface Z of degree n. Since $h^{n-2,0}(Z) = h^{0,n-2}(Z) = 1$, the result follows from Lemma 3.4. □

We are now able to solve the construction problem modulo m for the second outer Hodge numbers, i. e. the inner Hodge numbers $h^{p,q}$ with $p \in \{1, n-1\}$ or $q \in \{1, n-1\}$, via repeated blowups in smooth centres. By Serre duality, it is enough to consider the Hodge numbers $h^{p,q}$ with $(p,q) \in I_n$, where

$$I_n = \left\{ (1,q) \;\middle|\; q \in \{1, \ldots, n-1\} \right\} \cup \left\{ (p,1) \;\middle|\; p \in \{1, \ldots, n-1\} \right\}.$$

Corollary 3.15. *Let X be a smooth projective k-variety of dimension n. For any given collection of integers $(a^{p,q})_{(p,q)\in I_n}$ with $a^{n-1,1} = a^{1,n-1}$, there exists a birational morphism $\tilde{X} \to X$ obtained by a sequence of blowups in smooth centres such that*

$$h^{p,q}(\tilde{X}) \equiv a^{p,q} \pmod{m}$$

for all $(p,q) \in I_n$.

Proof. For $r \in \{1, \ldots, n-1\}$, let $b = a^{r,1} - h^{r,1}(X)$ and $c = a^{1,r} - h^{1,r}(X)$. We see that $b = c$ if $r = 1$ or $r = n-1$. Hence, we may apply Corollary 3.14 for all $r \in \{1, \ldots, n-1\}$ in descending order to obtain the result. □

Finally, we are ready to prove Theorem 3.2, which together with Proposition 3.10 implies our main result Theorem 3.1.

Proof of Theorem 3.2. We will proceed by induction on n. The case $n \leq 1$ is vacuous, as there are no inner Hodge numbers. Let $n \geq 2$, and assume the result is known in all dimensions $\leq n-1$. By Corollary 3.15, there exists a birational morphism $X_1 \to X$ obtained by a sequence of blowups in smooth centres such that for $(p,q) \in I_n$ we have

$$h^{p,q}(X_1) \equiv a^{p,q} - h^{p-1,q-1}\left(\mathbb{P}^{n-2}\right) \pmod{m}.$$

Let $X_2 \to X_1$ be the blowup in a point, and let $\mathbb{P}^{n-2} \subseteq X_2$ be a hyperplane in the exceptional divisor. By the induction hypothesis, there exists a birational morphism $\tilde{P} \to \mathbb{P}^{n-2}$ obtained by a sequence of blowups in smooth centres such that the Hodge numbers of $\tilde{P}$ are given by

$$h^{p,q}(\tilde{P}) \equiv \begin{cases} h^{p,q}\left(\mathbb{P}^{n-2}\right), & p \in \{0, n-2\} \text{ or } q \in \{0, n-2\}, \\ a^{p+1,q+1} - h^{p+1,q+1}(X_1), & \text{else.} \end{cases} \pmod{m}$$

Since $\tilde{P} \to \mathbb{P}^{n-2}$ is a sequence of blowups in smooth centres, we can blow up the (strict transforms of) the same centres in X_2 to get a birational morphism $X_3 \to X_2$ such that the strict transform of $\mathbb{P}^{n-2}$ is $\tilde{P}$. Blowing up $m-1$ more points coming from X_1 and applying the same construction in each of the exceptional divisors separately gives a birational morphism $X_4 \to X_1$ that does not change the Hodge numbers modulo m by the blowup formula of Lemma 3.4. Finally, if we let $\tilde{X} \to X_4$ be the blowup in one of the $\tilde{P}$ obtained in this way, we get

$$h^{p,q}(\tilde{X}) = h^{p,q}(X_1) + h^{p-1,q-1}(\tilde{P}) \equiv a^{p,q} \pmod{m}$$

for all (p,q) with $1 \leq p,q \leq n-1$, which finishes the induction step. $\square$

Remark 3.16. The proof above can be simplified if one assumes embedded resolution of singularities in arbitrary dimension. Indeed, by blowing up a finite number of points, we may assume that $h^{1,1}(X) \equiv a^{1,1} - 1 \pmod{m}$ and X contains $\mathbb{P}^{n-1}$. Now we claim that we can construct an $(n-2)$-dimensional subvariety Y in a blowup $X' \to X$ with $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$ such that $h^{p,q}(Y) \equiv a^{p+1,q+1} - h^{p+1,q+1}(X) \pmod{m}$. Then the blowup $\tilde{X} \to X'$ in Y has the required Hodge numbers.

To construct Y, first construct any smooth projective variety Z of dimension $n-2$ with the correct outer Hodge numbers using Proposition 3.10. Then Z is birational to a (possibly singular) hypersurface $Z' \subseteq \mathbb{P}^{n-1}$. Embedded resolution of $Z' \subseteq \mathbb{P}^{n-1}$ gives a birational map $X' \to X$ such that the strict transform of Z' is smooth, so Z' has the desired outer Hodge numbers by [CR11, Thm. 1]. By the induction hypothesis we may blow up further to get the inner Hodge numbers we want. Repeating this construction $m-1$ more times, as usual, gives $h^{p,q}(X') \equiv h^{p,q}(X) \pmod{m}$.

However, because resolution of singularities is currently unknown in positive characteristic beyond dimension 3, we have developed the above approach using embedded resolution of surfaces, Maruyama's theory of elementary transformations of projective bundles, and the fortuitous fact that the failure of Hodge symmetry is 'generated' by surfaces (see also [vDdB21, Thm. 2]).

Remark 3.17. Both the proof of Theorem 3.2 above (replacing Lemma 3.13 by an easy case of [PS19, Lem. 6]) and the alternative argument of Remark 3.16 using resolution of singularities give new methods to prove the characteristic zero result [PS19, Thm. 5].

Conversely, it is possible to adapt the methods of [PS19, §3] to prove Theorem 3.2, using the subvarieties from [PS19, Lem. 6] as well as projective bundles over the subvarieties from Lemma 3.13, but the analysis is a bit more intricate.

3.5. Polynomial relations

Corollary 3.18. *There are no polynomial relations among the Hodge numbers of smooth projective k-varieties of the same dimension besides the ones induced by Serre duality.*

Proof. Using [PS19, Lem. 8], this follows from Theorem 3.1 in the same way as [PS19, Cor. 3], except that we now consider the Hodge numbers $h^{p,q}$ with $0 \leq p \leq q \leq n$ and $(p, q) \neq (0, 0), (n, n)$. $\square$

Corollary 3.19. *There are no polynomial relations among the inner Hodge numbers of smooth projective k-varieties of any fixed birational equivalence class besides the ones induced by Serre duality.*

Proof. This follows from Theorem 3.2 in a similar fashion. $\square$

4. On the degree of algebraic cycles on hypersurfaces

Abstract

Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Griffiths and Harris conjectured in 1985 that the degree of every curve $C \subset X$ is divisible by d. Despite substantial progress by Kollár in 1991, this conjecture is not known for a single value of d. Building on Kollár's method, we prove this conjecture for infinitely many d, the smallest one being $d = 5005$. The set of these degrees d has positive density. We also prove a higher-dimensional analogue of this result and construct smooth hypersurfaces defined over $\mathbb{Q}$ that satisfy the conjecture.

This chapter is based on [Pau22].

4.1. Introduction

In their famous work [GH85], Griffiths and Harris made five conjectures about curves on a very general hypersurface $X \subset \mathbb{P}^4$ over $\mathbb{C}$. The weakest one is:

Conjecture 4.1 (Griffiths–Harris)**.** *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Then the degree of every curve $C \subset X$ is divisible by d.*

The conjecture was stated over $\mathbb{C}$, but could be considered over other fields as well. However, as we will see below, in the complex setting there is an intimate connection between this conjecture and the failure of the integral Hodge conjecture on X. Throughout the whole chapter, we thus work over the field of complex numbers, which was also the setting considered by Griffiths and Harris.

Conjecture 4.1 would follow from the stronger conjectures in [GH85] that C is algebraically or even rationally equivalent to a multiple of a plane section. Their strongest conjecture, stating that C is a complete intersection with a surface in $\mathbb{P}^4$, was disproven by Voisin [Voi89]. In contrast, Wu proved in [Wu90] that every curve $C \subset X$ of degree at most $2d-2$ is a complete intersection with a surface in $\mathbb{P}^4$. In particular, the degree of every curve $C \subset X$ is at least d. Nevertheless, Conjecture 4.1 is still open in any degree d.

More generally, one might conjecture:

Conjecture 4.2. *Let $n \geq 1$ be an integer, and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Then the degree of every positive-dimensional closed subvariety $Z \subset X$ is divisible by d.*

Note that this conjecture is wrong if $1 < d < 2n$ because a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d < 2n$ contains a line. The case $n = 1$ of this conjecture is trivial, and the case $n = 2$ follows from the Noether–Lefschetz theorem [Lef24]. For $n \geq 3$, however, Conjecture 4.2 is not known for a single d yet.

4.1.1. Kollár's method

For a very general hypersurface $X \subset \mathbb{P}^4$ of degree d, let us write

$$f_3(d) = \gcd\{\deg C \mid C \subset X \text{ curve}\} .$$

Conjecture 4.1 states that $f_3(d) = d$ for all $d \geq 6$.

The Trento examples [K+91], mostly due to Kollár, achieve substantial progress towards Conjecture 4.1 via specialization arguments. By degenerating a very general hypersurface $X \subset \mathbb{P}^4$ into a singular projection of a smooth projective threefold Y, the following results are obtained:

(1) $d \mid 6 \cdot f_3(d^3)$ for all $d \geq 1$ (Kollár, see also [SV05, section 2])
(2) $d \mid 6 \cdot f_3(3d^2)$ for all $d \geq 4$ (Kollár)
(3) $d \mid 2 \cdot f_3(6d)$ for all $d \geq 9$ (van Geemen, improved by [DHS94])

These results naturally generalize to arbitrary dimension $n \geq 3$ (see section 4.2), but they do not prove Conjecture 4.2 for any $d \geq 2n$.

4.1.2. Main results

The main purpose of this chapter is to show the following:

Theorem 4.3. *Let $n \geq 3$ be an integer. Then there exists a set of degrees d with positive density such that Conjecture 4.2 is true in degree d.*

In particular, the case $n = 3$ proves the conjecture of Griffiths and Harris for infinitely many degrees d. The smallest of them is $d = 5005$. Hence, $d = 5005$ is the first degree where Conjecture 4.1 is currently known.

For a projective variety X, let us introduce the group

$$Z^{2c}(X) = \frac{H^{c,c}(X, \mathbb{Z})}{\langle \text{alg. classes} \rangle} ,$$

which measures the failure of the integral Hodge conjecture on X in codimension c. As a consequence of Theorem 4.3, we get:

Corollary 4.4. *Let $n \geq 3$ be an integer. Then there exists a set of degrees d with positive density such that*

$$Z^{2c}(X) \cong \mathbb{Z}/d$$

for a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d and for $\frac{n}{2} < c < n$.

In particular, the integral Hodge conjecture fails for very general hypersurfaces $X \subset \mathbb{P}^{n+1}$ of these degrees d.

For $n = 3$, the previous result from [DHS94] (item (3) above) allows to disprove the integral Hodge conjecture for a set of degrees with density $\frac{1}{6}$. With our approach, we can actually show the failure of the integral Hodge conjecture for a set of degrees with density 1:

Theorem 4.5. *Let $n \geq 3$ be an integer. Then there exists a set of degrees d with density 1 such that the integral Hodge conjecture for very general hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d is false in every codimension c with $\frac{n}{2} < c < n$.*

Theorems 4.3 and 4.5 as well as the known results (1), (2), and (3) concern very general hypersurfaces. This might possibly exclude all hypersurfaces defined over number fields. However, Totaro proved in [Tot13] that (1), (2), and (3) are in most cases also valid for certain smooth hypersurfaces $X \subset \mathbb{P}^4$ defined over $\mathbb{Q}$. Using Totaro's results, we show:

Theorem 4.6. *There exists a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$ defined over $\mathbb{Q}$ such that the degree of every curve $C \subset X_{\overline{\mathbb{Q}}}$ is divisible by d.*

For example, one can choose $d = 7 \cdot 13 \cdot 19 \cdot 31 = 53599$ (see Proposition 4.20).

4.1.3. General observations and notation

Let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree d and let $\alpha = c_1(\mathcal{O}_X(1)) \in H^{1,1}(X, \mathbb{Z})$ denote the hyperplane class.

By the Lefschetz hyperplane theorem, the restriction map

$$H^i(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^i(X, \mathbb{Z})$$

is an isomorphism for $i < n$. Therefore,

$$H^{2c}(X, \mathbb{Z}) = H^{c,c}(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha^c \quad \text{for } c < \tfrac{n}{2}.$$

Hence, Conjecture 4.2 is true for subvarieties $Z \subset X$ of codimension $c < \frac{n}{2}$. Moreover, $Z^{2c}(X)$ is trivial for $c < \frac{n}{2}$.

For $\frac{n}{2} < c \leq n$, Poincaré duality implies that

$$H^{2c}(X, \mathbb{Z}) = H^{c,c}(X, \mathbb{Z}) = \mathbb{Z} \cdot \frac{1}{d}\alpha^c \quad \text{for } \tfrac{n}{2} < c \leq n.$$

Therefore, no topological obstructions on the degree of subvarieties $Z \subset X$ of codimension c with $\frac{n}{2} < c \leq n$ exist. In this case, we can rephrase Conjecture 4.2 in terms of the group $Z^{2c}(X)$. Since α^c is clearly algebraic, $Z^{2c}(X)$ is a quotient of $\mathbb{Z}/d$. The order of $Z^{2c}(X)$ is given by the greatest common divisor of the degrees of all subvarieties $Z \subset X$ of codimension c. Hence, Conjecture 4.2 in codimension c for $\frac{n}{2} < c < n$ is equivalent to $Z^{2c}(X) \cong \mathbb{Z}/d$. In particular, Theorem 4.3 implies Corollary 4.4.

Note that it suffices to prove Conjecture 4.2 for curves $C \subset X$ because every positive-dimensional subvariety $Z \subset X$ gives rise to a curve $C \subset X$ of the same degree after intersecting Z with a suitable linear subspace.

If $X \subset \mathbb{P}^{n+1}$ is very general, the order of the group $Z^{2n-2}(X)$ only depends on n and d. We set $f_n(d) := |Z^{2n-2}(X)|$ for $n \geq 3$ and $d \geq 1$. In other words,

$$f_n(d) = \gcd\{\deg C \mid C \subset X \text{ curve}\}\ .$$

For $n = 3$, this agrees with the definition of $f_3(d)$ given earlier.

We know that $f_n(d) \mid d$ for all degrees d, and Conjecture 4.2 in degree d is equivalent to $f_n(d) = d$.

4.1.4. Proof idea and overview

Looking at the existing results towards Conjecture 4.1, statement (3) seems to be more powerful than (1) and (2). However, the main idea for proving Theorem 4.3 is to combine (1), (2), and (3) in order to show $d \mid f_3(d)$ for certain degrees d. This is based on the observation that $d \mid f_n(d_1)$ and $d \mid f_n(d_2)$ together imply $d \mid f_n(d_1 + d_2)$, as can be seen by another degeneration argument.

In section 4.2, we develop higher-dimensional analogues of the Trento examples [K+91]. These allow to carry out our approach in arbitrary dimension $n \geq 3$.

In section 4.3, we will prove the following statement behind Theorem 4.3:

Proposition 4.7. *Let $n \geq 3$ be an integer. Then we have $f_n(d) = d$ if d is coprime to $n!$ and the largest prime power q dividing d satisfies*

$$\left(\binom{n}{2} - 1\right) \cdot q^n + \left(n! - \binom{n}{2}\right) \cdot q^{n-1} + (2^n + 1) \cdot n! \leq d\,.$$

For $n = 3$, the smallest degree d with this property is

$$d = 5 \cdot 7 \cdot 11 \cdot 13 = 5005\,.$$

In section 4.4, we will see that the positive integers d fulfilling the condition in Proposition 4.7 have positive density for all $n \geq 3$, thus completing the proof of Theorem 4.3.

Finally, we prove Theorem 4.5 in section 4.5 and Theorem 4.6 in section 4.6.

4.2. The Trento examples

In this section, we take a closer look at the Trento examples from [K+91] by generalizing them to arbitrary dimension $n \geq 3$.

All examples rely on the following lemma:

Lemma 4.8 (Kollár)**.** *Let $n \geq 3$ be an integer. Suppose that there exists a smooth projective variety Y of dimension n with a very ample line bundle L such that $L^n = d$ and $k \mid B \cdot L$ for every curve $B \subset Y$. Then we have*

$$k \mid n! \cdot f_n(d)\,.$$

Proof. We consider the embedding $Y \subset \mathbb{P}^N$ given by the very ample line bundle L, and take a general linear projection

$$\pi\colon Y \to \mathbb{P}^{n+1} .$$

Then $\pi(Y) \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $L^n = d$.

By [Mat73] (see also [BE10]), each fiber of π has at most $n+1$ distinct points (note that the fibers of π might have a much larger degree than $n+1$ due to their non-reduced scheme structure, but for our argument we only need that they consist of at most $n+1$ points topologically). Moreover, only finitely many fibers have exactly $n+1$ distinct points. Hence, for every curve $C \subset \pi(Y)$, the curve $B = \pi^{-1}(C)^{\mathrm{red}} \subset Y$ admits a finite surjective map $\pi|_B\colon B \to C$ of degree at most n. Therefore, we have $B \cdot L \mid n! \cdot \deg C$ and thus $k \mid n! \cdot \deg C$.

Now if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree d, every curve on X specializes to a curve $C \subset \pi(Y)$ of the same degree. For more details on this degeneration argument, see [SV05, section 2]. Since $k \mid n! \cdot \deg C$, it follows that $k \mid n! \cdot f_n(d)$. □

Corollary 4.9 (Kollár). *For all $n \geq 3$ and $d \geq 1$, we have*

$$d \mid n! \cdot f_n(d^n) .$$

In particular, we have $d \mid f_n(d^n)$ if d is coprime to $n!$.

Proof. We apply Lemma 4.8 to $Y = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(d)$. □

Corollary 4.10 (Kollár). *For all $n \geq 3$ and $d \geq 4$, we have*

$$d \mid n! \cdot f_n\left(\tbinom{n}{2} d^{n-1}\right) .$$

In particular, we have $d \mid f_n\left(\binom{n}{2} d^{n-1}\right)$ if d is coprime to $n!$.

Proof. We take $Y = S \times \mathbb{P}^{n-2}$ where $S \subset \mathbb{P}^3$ is a very general surface of degree $d \geq 4$. On Y, we consider the very ample line bundle

$$L = \mathrm{pr}_1^* \mathcal{O}_S(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^{n-2}}(d) .$$

Then we have $L^n = \binom{n}{2} d^{n-1}$ (note that the factor $\binom{n}{2}$ is accidentally missing in [K$^+$91]). If $B \subset Y$ is a curve, we obtain

$$B \cdot L = (\mathrm{pr}_1)_* B \cdot \mathcal{O}_S(1) + (\mathrm{pr}_2)_* B \cdot \mathcal{O}_{\mathbb{P}^{n-2}}(d) \equiv 0 \pmod{d} ,$$

because d divides the degree of the curve $\mathrm{pr}_1(B) \subset S$ by the Noether–Lefschetz theorem [Lef24]. Therefore, Lemma 4.8 implies the result. □

Corollary 4.11 (van Geemen, Debarre–Hulek–Spandaw)**.** *For all $n \geq 3$ and $d \geq 2^n + 1$, we have*

$$d \mid (n-1)! \cdot f_n(n! \cdot d)\ .$$

In particular, we have $d \mid f_n(n! \cdot d)$ if d is coprime to $(n-1)!$.

Proof. Let (Y, L) be a very general polarized Abelian variety of dimension n and type $(1, \ldots, 1, d)$. Then we have $L^n = n! \cdot d$. It was shown in [DHS94] that the line bundle L is very ample.

Since L is of type $(1, \ldots, 1, d)$, we have

$$c_1(L) = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \cdots + \mathrm{d}x_{2n-3} \wedge \mathrm{d}x_{2n-2} + d \cdot \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n} \in H^2(Y, \mathbb{Z})$$

for a suitable basis $\mathrm{d}x_1, \ldots, \mathrm{d}x_{2n}$ of $H^1(Y, \mathbb{Z})$. From this we see that

$$\frac{c_1(L)^{n-1}}{(n-1)!} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_{2n-3} \wedge \mathrm{d}x_{2n-2} + d \cdot \ldots \in H^{2n-2}(Y, \mathbb{Z})$$

is not divisible by any integer larger than 1. If Y is very general, the algebraic classes in $H^{2n-2}(Y, \mathbb{Z})$ are rational multiples of $c_1(L)^{n-1}$, and thus integral multiples of $c_1(L)^{n-1}/(n-1)!$. Therefore, the degree of every curve $B \subset Y$ is divisible by $L^n/(n-1)! = nd$. Hence, Lemma 4.8 gives $nd \mid n! \cdot f_n(n! \cdot d)$. □

4.3. Proof of Proposition 4.7

We combine Corollaries 4.9, 4.10, and 4.11 via the following simple observation:

Lemma 4.12. *If $d \mid f_n(d_1)$ and $d \mid f_n(d_2)$, then $d \mid f_n(d_1 + d_2)$.*

Proof. Let $C \subset X$ be a curve on a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d_1 + d_2$. By the same degeneration argument which we used in the proof of Lemma 4.8, *every* hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d_1 + d_2$ contains a curve of the same degree as C. In particular, we can choose $X = X_1 \cup X_2$ to be the union of very general hypersurfaces $X_1, X_2 \subset \mathbb{P}^{n+1}$ of degrees d_1 and d_2, respectively. Then every irreducible component of a curve $C \subset X_1 \cup X_2$ lies on X_1 or X_2. By assumption, the degrees of these components are divisible by d. We conclude that $d \mid \deg C$. □

Now we can prove Proposition 4.7.

Proof of Proposition 4.7. Since d is a product of pairwise coprime powers of primes, it suffices to show $q \mid f_n(d)$ for every prime power $q \mid d$. By assumption, we have

$$(\binom{n}{2} - 1) \cdot q^n + (n! - \binom{n}{2}) \cdot q^{n-1} + (2^n + 1) \cdot n! \leq d \,. \tag{4.1}$$

We choose $i \in \{0, \ldots, \binom{n}{2} - 1\}$ such that

$$d \equiv i \cdot q^n \pmod{\binom{n}{2}} \,.$$

This is possible because $\binom{n}{2}$ divides $n!$ and q is coprime to $n!$.

Then we choose $j \in \{0, \ldots, n! - 1\}$ such that

$$d \equiv i \cdot q^n + j \cdot q^{n-1} \pmod{n!} \,.$$

Our choice of i implies that j is divisible by $\binom{n}{2}$, so we have $j \leq n! - \binom{n}{2}$.

By our choice of j, there exists an integer k such that

$$d = i \cdot q^n + j \cdot q^{n-1} + k \cdot n! \,.$$

Since $q \mid d$, we have $q \mid k$. And from (4.1) we get $k \geq 2^n + 1$.

Now we have:

- $q \mid f_n(q^n)$ by Corollary 4.9
- $q \mid f_n(\binom{n}{2} q^{n-1})$ by Corollary 4.10
- $k \mid (n-1)! \cdot f_n(k \cdot n!)$ by Corollary 4.11 and thus $q \mid f_n(k \cdot n!)$

Combining these results via repeated usage of Lemma 4.12, we obtain

$$q \mid f_n\left(i \cdot q^n + \frac{j}{\binom{n}{2}} \cdot \binom{n}{2} q^{n-1} + k \cdot n!\right) = f_n(d) \,. \qquad \square$$

Remark 4.13. Using only Corollaries 4.9 and 4.11, one can show a weaker statement where (4.1) is replaced by the assumption

$$(n! - 1) \cdot q^n + (2^n + 1) \cdot n! \leq d \,.$$

It turns out that this stronger condition results in a set of degrees d with the same density. Therefore, Corollary 4.10 is not strictly necessary to obtain Theorem 4.3. However, only together with Corollary 4.10 we can prove Conjecture 4.1 for $d = 5005$.

4.4. Some analytic number theory

A set A of positive integers has *positive density* if

$$\liminf_{m\to\infty} \frac{|A \cap \{1,\dots,m\}|}{m} > 0 \, .$$

In this section, we want to prove the following:

Proposition 4.14. *Let $n \geq 1$ be an integer and $\lambda > 0$ a real number. Then the positive integers d coprime to $n!$ such that the largest prime power dividing d is not larger than $\lambda \cdot d^{1/n}$ have positive density.*

Together with Proposition 4.7, this will complete the proof of Theorem 4.3, since for $d \gg 0$ the condition $q \leq \binom{n}{2}^{-1/n} d^{1/n}$ implies

$$\left(\binom{n}{2} - 1\right) \cdot q^n + \left(n! - \binom{n}{2}\right) \cdot q^{n-1} + (2^n + 1) \cdot n! \leq d \, ,$$

so Proposition 4.7 applies to a set of degrees d with positive density.

We use the following easy lemma on the distribution of prime powers:

Lemma 4.15. *Let $\Pi(m)$ denote the number of prime powers $\leq m$. Then*

$$\frac{\Pi(m)}{m} \overset{m\to\infty}{\longrightarrow} 0 \, .$$

Proof. By the prime number theorem, we have

$$\frac{\pi(m)}{m} \overset{m\to\infty}{\longrightarrow} 0 \, ,$$

where $\pi(m)$ counts the prime numbers $\leq m$. Now if $p^e \leq m$ is a prime power with $e \geq 2$, we have $e \leq \log_2 m$ and $p \leq \sqrt{m}$, so we conclude by noting that

$$\frac{\log_2 m \cdot \sqrt{m}}{m} \overset{m\to\infty}{\longrightarrow} 0 \, . \qquad \square$$

We also need the following consequence of Mertens' theorem:

Lemma 4.16. *We have*

$$\sum_{x^{1/n} < p \leq x} \frac{1}{p} \overset{x\to\infty}{\longrightarrow} \log n \, ,$$

where the sum runs only over prime numbers p.

Proof. By [Mer74], there exists a constant C such that

$$\sum_{p \le x} \frac{1}{p} - \log\log x \overset{x\to\infty}{\longrightarrow} C \; .$$

Since $\log\log x - \log\log x^{1/n} = \log n$, we conclude that

$$\sum_{x^{1/n} < p \le x} \frac{1}{p} = \sum_{p \le x} \frac{1}{p} - \sum_{p \le x^{1/n}} \frac{1}{p} \overset{x\to\infty}{\longrightarrow} \log n \; . \qquad \square$$

Now we can prove Proposition 4.14.

Proof of Proposition 4.14. Let $\alpha > 1$ be a real number. Dickman proved in [Dic30] that the positive integers d whose largest prime divisor is not larger than $d^{1/\alpha}$ have density $\rho(\alpha)$, where ρ denotes the Dickman function. More generally, this result was proven for arithmetic progressions in [Buc49]. Therefore, the positive integers d coprime to $n!$ whose largest prime divisor is not larger than $d^{1/\alpha}$ have density

$$\frac{\varphi(n!)}{n!} \cdot \rho(\alpha) \; ,$$

where φ denotes Euler's totient function. Since ρ is continuous, it follows that the positive integers d coprime to $n!$ whose largest prime divisor is not larger than $\lambda \cdot d^{1/n}$ have density $\frac{\varphi(n!)}{n!} \cdot \rho(n) > 0$.

In other words, Proposition 4.14 holds if we replace 'prime power' by 'prime number'. Hence, it suffices to show that the positive integers d divisible by a prime power $q = p^e > \lambda \cdot d^{1/n}$ with $e \ge 2$ have density 0.

For a given x, let us consider the number $N(x)$ of positive integers $d \le x$ with this property. Any such d can be written as

$$d = q \cdot r \; ,$$

where $q = p^e \ge \lambda \cdot d^{1/n}$ is a prime power with $e \ge 2$. For fixed $q \le x^{1/n}$, there are at most $\lambda^{-n} q^{n-1}$ possibilities for d because

$$r = \frac{d}{q} \le \frac{\lambda^{-n} q^n}{q} = \lambda^{-n} q^{n-1} \; .$$

For fixed $q > x^{1/n}$, there are at most $\frac{x}{q}$ possibilities for d because

$$r = \frac{d}{q} \le \frac{x}{q} \; .$$

Together we obtain the upper bound

$$N(x) \;\leq\; \lambda^{-n} \cdot \sum_{q \leq x^{1/n}} q^{n-1} \;+\; x \cdot \sum_{x^{1/n} < q \leq x} \frac{1}{q} \,,$$

where both sums run only over prime powers $q = p^e$ with $e \geq 2$.

Using Lemma 4.15, we get

$$\frac{1}{x} \cdot \sum_{q \leq x^{1/n}} q^{n-1} \;\leq\; \frac{1}{x} \cdot \Pi(x^{1/n}) \cdot \left(x^{1/n}\right)^{n-1} = \frac{\Pi(x^{1/n})}{x^{1/n}} \xrightarrow{x\to\infty} 0 \,,$$

so in order to prove $\frac{N(x)}{x} \to 0$ for $x \to \infty$, it remains to show that

$$\sum_{x^{1/n} < q \leq x} \frac{1}{q} \xrightarrow{x\to\infty} 0 \,.$$

For $q = p^e \leq x$, we have $e \leq \log_2 x$ and thus

$$\begin{aligned} \limsup_{x\to\infty} \sum_{x^{1/n} < q \leq x} \frac{1}{q} &= \limsup_{x\to\infty} \sum_{e=2}^{\lfloor \log_2 x \rfloor} \sum_{x^{1/n} < p^e \leq x} \frac{1}{p^e} \\ &\leq \limsup_{x\to\infty} \left(\sum_{e=2}^{n} \sum_{x^{1/en} < p \leq x^{1/e}} \frac{x^{-\frac{e-1}{en}}}{p} + \sum_{e=n+1}^{\lfloor \log_2 x \rfloor} \frac{x^{1/e}}{x^{1/n}} \right) \,. \end{aligned}$$

Here we used $\frac{1}{p^e} = \frac{p^{-(e-1)}}{p} \leq \frac{x^{-\frac{e-1}{en}}}{p}$ for $p > x^{1/en}$ if $2 \leq e \leq n$, and $\frac{1}{p^e} \leq \frac{1}{x^{1/n}}$ for $p > x^{1/en}$ if $e \geq n+1$. Applying Lemma 4.16 for each $2 \leq e \leq n$, and using $x^{1/e} \leq x^{1/(n+1)}$ for $e \geq n+1$, we obtain

$$\cdots \leq \limsup_{x\to\infty} \left(\sum_{e=2}^{n} \frac{\log n}{x^{\frac{e-1}{en}}} + \frac{\log_2 x}{x^{\frac{1}{n(n+1)}}} \right) = 0 \,. \qquad \square$$

Remark 4.17. The proof shows that the density in Theorem 4.3 amounts to

$$\frac{\varphi(n!)}{n!} \cdot \rho(n) \,.$$

For example, the density for $n = 3$ is $\frac{1}{3} \cdot \rho(3) \approx 1.6\%$.

4.5. Failure of the integral Hodge conjecture

By the work of Kollár [K+91], hypersurfaces provide an example for varieties where the integral Hodge conjecture fails due to a non-torsion cohomology class. Theorem 4.5 says that this counterexample works for almost all degrees d (in the sense of density).

Proof of Theorem 4.5. The failure of the integral Hodge conjecture in degree d is equivalent to $f_n(d) \neq 1$. Hence, we need to show that the positive integers d with $f_n(d) \neq 1$ have density 1. If d has a prime divisor p coprime to $n!$ such that

$$\left(\binom{n}{2} - 1\right) \cdot p^n + \left(n! - \binom{n}{2}\right) \cdot p^{n-1} + (2^n + 1) \cdot n! \leq d \, ,$$

then the proof of Proposition 4.7 shows that $p \mid f_n(d)$. Therefore, for every prime $p > n$ all sufficiently large multiples d of p satisfy $f_n(d) \neq 1$.

For a given $\varepsilon > 0$, we can find distinct primes $p_1, \ldots, p_N > n$ such that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_N} > \frac{1}{\varepsilon}$$

since the sum of the reciprocals of all primes diverges. We know from the previous paragraph that for $d \gg 0$, we might have $f_n(d) = 1$ only if d is not divisible by any of the primes $p_1, \ldots, p_N$. Hence, the density of these d is at most

$$\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_N}\right) < \frac{1}{\left(1 + \frac{1}{p_1}\right) \cdots \left(1 + \frac{1}{p_N}\right)} < \frac{1}{\frac{1}{p_1} + \cdots + \frac{1}{p_N}} < \varepsilon \, .$$

This concludes the proof. $\square$

Remark 4.18. For any degree $d \geq 1$, there exist special smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree d which do satisfy the integral Hodge conjecture in every codimension c with $\frac{n}{2} < c < n$. For example, we can take the Fermat hypersurface

$$\{x_0^d + \cdots + x_{n+1}^d = 0\} \subset \mathbb{P}^{n+1} \, ,$$

since it contains an $(n - c)$-dimensional linear subspace for any $\frac{n}{2} < c < n$.

Remark 4.19. There are infinitely many degrees d for which we are not able to disprove the integral Hodge conjecture. In particular, this problem remains open when d is a prime number, in which case the failure of the integral Hodge conjecture is equivalent to Conjecture 4.2.

4.6. Example over $\mathbb{Q}$

The basic idea in [Tot13] is to replace the original degeneration arguments with degenerations to positive characteristic. To prove Theorem 4.6, we apply this idea to the proof of Proposition 4.7 and use some of Totaro's results.

Proposition 4.20. *There exists a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d = 7 \cdot 13 \cdot 19 \cdot 31 = 53599$ defined over $\mathbb{Q}$ such that the degree of every curve $C \subset X_{\overline{\mathbb{Q}}}$ is divisible by d.*

Note that every curve on $X_{\mathbb{C}}$ specializes to a curve $C \subset X_{\overline{\mathbb{Q}}}$ of the same degree (by viewing $\mathbb{C}$ as the algebraic closure of a purely transcendental field extension of $\overline{\mathbb{Q}}$), hence $X_{\mathbb{C}}$ satisfies the conclusion of the conjecture of Griffiths and Harris.

Proof of Proposition 4.20. We will show the following lemma:

Lemma 4.21. *Let q be any of the four prime divisors of d. Then there exists a prime p and a hypersurface $Y \subset \mathbb{P}^4_{\mathbb{F}_p}$ of degree d such that the degree of every curve $C \subset Y_{\overline{\mathbb{F}_p}}$ is divisible by q. Moreover, p can be chosen such that finitely many given primes are avoided.*

Once this lemma is proven, we proceed as follows: We ensure that the primes p for each prime divisor $q \mid d$ are pairwise different. Then we use the Chinese remainder theorem to construct a smooth hypersurface $X \subset \mathbb{P}^4$ defined over $\mathbb{Q}$ that simultaneously specializes to all four hypersurfaces $Y \subset \mathbb{P}^4_{\mathbb{F}_p}$ from Lemma 4.21. This X satisfies our claim. □

Proof of Lemma 4.21. Since $q \equiv 1 \pmod 6$, we can write $d = q^3 + 6k$ for some integer k. Note that $k \geq 38$. As in the proof of Lemma 4.12, we want to take $Y = Y_1 \cup Y_2$, where $Y_1, Y_2 \subset \mathbb{P}^4_{\mathbb{F}_p}$ are two hypersurfaces of degrees q^3 and $6k$, respectively, such that every curve $C \subset (Y_i)_{\overline{\mathbb{F}_p}}$ has degree divisible by q.

We first construct Y_1. A priori, [Tot13, Corollary 4.2] only gives hypersurfaces $Y_1 \subset \mathbb{P}^4$ over $\overline{\mathbb{F}_p}$ with this property for every $p > q^3$. However, as in the proofs of [Tot13, Lemma 5.1] and [Tot13, Theorem 6.1], we can apply [Tot13, Lemma 4.3] to $\mathbb{P}^3_{\mathbb{Q}}$ (polarized by $\mathcal{O}_{\mathbb{P}^3}(q)$) to get a rational map to $\mathbb{P}^4_{\mathbb{Z}}$ and obtain hypersurfaces $Y_1 \subset \mathbb{P}^4$ over $\mathbb{F}_p$ after excluding finitely many primes p.

Now we construct Y_2. The proof of [Tot13, Theorem 6.1] yields a prime p and a hypersurface $Y_2 \subset \mathbb{P}^4_{\mathbb{F}_p}$ of degree $6k$ such that $k \mid 6 \cdot \deg C$ for every curve $C \subset (Y_2)_{\overline{\mathbb{F}_p}}$. Since k is a multiple of q and q is coprime to 6, it follows that $q \mid \deg C$. Furthermore, we can guarantee that p is different from finitely many given primes (including also the primes where the construction of Y_1 does not work) by doing the argument of [Tot13, Theorem 6.1] over $\mathbb{Z}[1/P]$ instead of $\mathbb{Z}$, where P is the product of these finitely many primes. □

Remark 4.22. For simplicity, we gave only one specific example over $\mathbb{Q}$. The above argument obviously works for other values than $d = 53599$ as well.

5. On a unirational counterexample to the integral Hodge conjecture

Abstract

Schreieder constructed the first example of a unirational fourfold where the integral Hodge conjecture fails in codimension 2. We take a new look at his example and describe it from a different, more geometric perspective using Borel–Moore homology. Our approach is aimed towards deciding whether the non-algebraic Hodge class on this variety is a torsion class.

5.1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$. The integral Hodge conjecture in codimension k says that the cycle class map

$$\mathrm{CH}^k(X) \to H^{k,k}(X,\mathbb{Z}) = \{\mu \in H^{2k}(X,\mathbb{Z}) \mid \mu_{\mathbb{C}} \in H^{k,k}(X)\}$$

is surjective. This is true for $k = 1$ by the Lefschetz $(1,1)$ theorem.

After Atiyah and Hirzebruch [AH61] constructed the first counterexamples to this statement (for $k = 2$), the failure of the integral Hodge conjecture has been extensively studied for different classes of algebraic varieties, see e. g. [K$^+$91, Tot97, SV05, CTV12, Tot13, Sch19, BO20, OS20, BW20a, BW20b, Dia20]. There are two fundamentally different ways how the integral Hodge conjecture can fail:

(1) There exists a non-algebraic class $\mu \in H^{2k}(X,\mathbb{Z})$ such that $m \cdot \mu = 0$ for some positive integer m.

(2) There exists a non-algebraic class $\mu \in H^{2k}(X,\mathbb{Z})$ such that $m \cdot \mu$ is algebraic for some positive integer m, but $m \cdot (\mu + \delta) \neq 0$ for all algebraic classes $\delta \in H^{2k}(X,\mathbb{Z})$ and all positive integers m.

Assuming the (rational) Hodge conjecture is true, every counterexample arises in one of these two ways. Let us say that counterexamples of the form (1) are of *torsion type* and those of the form (2) are of *non-torsion type*.

The original counterexample in [AH61] was of torsion type. The first non-torsion counterexample was found by Kollár [K+91] and is given by very general hypersurfaces $X \subset \mathbb{P}^4$ of certain degrees $d \geq 6$ (the working values for d have density 1, see [Pau22] or chapter 4), where we have $H^4(X,\mathbb{Z}) \cong \mathbb{Z}$. These threefolds are of general type, which leads to the question whether the integral Hodge conjecture is true for threefolds of Kodaira dimension < 3.

Indeed, this is the case for Kodaira dimension $-\infty$ (i. e. for uniruled threefolds), as shown by Voisin [Voi06]. In the same paper, she proved the integral Hodge conjecture for threefolds X with $\omega_X \cong \mathcal{O}_X$ and $H^2(X,\mathcal{O}_X) = 0$. Grabowski [Gra04] gave a proof for abelian threefolds. Totaro [Tot21] generalized the previous two results to all threefolds X of Kodaira dimension 0 with $H^0(X,\omega_X) \neq 0$. In contrast, Benoist and Ottem [BO20] constructed counterexamples for any Kodaira dimension ≥ 0.

One commonly introduces the group

$$Z^{2k}(X) = \mathrm{Coker}(\mathrm{CH}^k(X) \to H^{k,k}(X,\mathbb{Z}))$$

to measure the failure of the integral Hodge conjecture in codimension k. Since $Z^4(X)$ is easily seen to be a birational invariant, it is an interesting question whether there exist rationally connected or even unirational counterexamples to the integral Hodge conjecture in codimension 2. By the aforementioned result of Voisin [Voi06], such examples would need to have dimension at least 4.

Colliot-Thélène and Voisin [CTV12] gave a description of $Z^4(X)$ in terms of unramified cohomology. If $\mathrm{CH}_0(X)$ is supported on a surface, we have

$$Z^4(X)[m] \cong H^3_{\mathrm{nr}}(X,\mathbb{Z}/m)$$

for all integers $m \geq 2$. Using this, it follows that the integral Hodge conjecture fails for a unirational sixfold constructed by Colliot-Thélène and Ojanguren [CTO89]. Schreieder [Sch19] was finally able to construct such unirational counterexamples in the smallest possible dimension 4.

Since Schreieder's argument relies on the abstract description of $Z^4(X)[2]$ via unramified cohomology, it is unclear to which of the two possible categories his counterexample belongs, i. e. whether it is of torsion or non-torsion type. In this chapter, we carry out the first steps towards answering this question. To this end, we perform a careful geometric study of the non-zero class in $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2)$ constructed by Schreieder.

In [CTV12], the following question is asked:

> Des modèles lisses des variétés considérées étant difficiles à construire, les éléments non triviaux correspondants du groupe $Z^4(X)$, c'est-à-dire des classes de Hodge entières non algébriques, sont difficiles à analyser. Ces classes proviennentelles, comme c'est le cas dans les exemples d'Atiyah–Hirzebruch, de classes de torsion dans $H^4(X, \mathbb{Z})$?

In English:

> Since smooth models of the considered varieties are difficult to construct, the corresponding non-trivial elements of the group $Z^4(X)$, i. e. the non-algebraic integral Hodge classes, are difficult to analyse. Do these classes, as in the Atiyah–Hirzebruch examples, originate from torsion classes in $H^4(X, \mathbb{Z})$?

While we focus on the fourfold from [Sch19] here, our viewpoint is in principle also applicable to the sixfold from [CTO89].

Schreieder's counterexample is described by a singular birational model, which is a conic bundle over $\mathbb{P}^3$. As remarked by Colliot-Thélène and Voisin, one difficulty in answering the above question lies in the construction of a smooth resolution. With a small trick, we can compare the conic bundle to a much simpler one that admits a smooth resolution by a hypersurface in a multiweighted projective space. Although the simplified bundle is not birational to the original one, they agree on a subset on which a natural representative of the considered unramified cohomology class is supported.

Arguments with unramified cohomology are usually rather abstract. In this chapter, we follow a different approach: Using the setup of [Sch23], we identify unramified cohomology classes with their duals in Borel–Moore homology. This allows us to work with them very concretely, e. g. we can explicitly provide real submanifolds representing unramified cohomology classes. By [Sch23, Theorem 7.7], the question whether the corresponding non-algebraic integral Hodge class is torsion reduces to the problem whether Schreieder's class in $H^3_{\mathrm{nr}}(X, \mathbb{Z}/2)$ extends to a global class in $H^3(X, \mathbb{Z}/2)$. Our more analytic/geometric approach could be compared in style to [AM72, § 2]. There, the easier case of $H^2_{\mathrm{nr}}(X, \mathbb{Z}/2)$ is investigated, where X is birational to a certain conic bundle over $\mathbb{P}^2$.

Along the lines, we obtain a geometric description of the map

$$K_i^M(\mathbb{C}(X)) \to H^i(\mathbb{C}(X), \mathbb{Z})$$

from Milnor K-theory to Galois cohomology, which might be of independent interest. As this description shows, a given symbol $\{f_1, \dots, f_i\}$ actually induces a well-defined class in $H^i(U, \mathbb{Z})$, where $U \subset X$ is the complement of the zeros and poles of the rational functions $f_1, \dots, f_i$.

After fixing canonical representatives for the Borel–Moore homology classes dual to $(f) \in H^1(U, \mathbb{Z})$ for $f \in \mathbb{C}(X)^*$, an essential preparation for the later arguments constitutes of geometric descriptions of certain Borel–Moore chains whose boundaries "explain" the relations $(fg) = (f) + (g)$ and $(f) \cup (1 - f) = 0$ from Milnor K-theory. This is done in section 5.2.

In section 5.3, we construct a smooth resolution of Schreieder's conic bundle. This is done such that we can often work on the resolution of a much simpler conic bundle later. In section 5.4, we carry out the crucial computation in Borel–Moore homology which geometrically explains the unramifiedness of the considered class. This allows to give an explicit description of an algebraic cycle representing twice the non-algebraic Hodge class in this counterexample (see Corollary 5.11).

The question whether Schreieder's counterexample is of torsion type remains open. We hope that the results in this chapter help to answer this question in the future.

5.2. Unramified cohomology and Borel–Moore homology

Many equivalent definitions of unramified cohomology exist, see e. g. [CT95, Theorem 4.1.1]. In the following, we adhere to a new viewpoint introduced in [Sch23], which turns out to be very fruitful.

Let X be a smooth projective variety over $\mathbb{C}$. For an abelian group A, the direct limit

$$H^i(\mathbb{C}(X), A) := \varinjlim_{U} H^i(U, A)$$

over all Zariski open subsets $\emptyset \neq U \subset X$ agrees with the Galois cohomology of $\mathbb{C}(X)/\mathbb{C}$. Here, $H^i(U, A)$ denotes singular cohomology with respect to the analytic topology. This interplay between the Zariski topology and the analytic topology on X will play an important role.

The subgroup $H^i_{\mathrm{nr}}(X, A) \subset H^i(\mathbb{C}(X), A)$ consists by definition of the elements which can be represented by a cohomology class on a *big* open subset, i. e. a Zariski open subset $U \subset X$ such that $\operatorname{codim}_X(X \setminus U) \geq 2$. In other words,

$$H^i_{\mathrm{nr}}(X, A) := \operatorname{Im}\left(H^i(F_1X, A) \to H^i(F_0X, A)\right)$$

where $H^i(F_jX, A)$ denotes the direct limit of $H^i(U, A)$ over all Zariski open subsets $U \subset X$ such that $\operatorname{codim}_X(X \setminus U) > j$, see [Sch23, Definition 5.1].

The obstruction for extending a class in $H^i(\mathbb{C}(X), A)$ over the generic point of a prime divisor $D \subset X$ is given by the residue morphism $\partial_D \colon H^i(\mathbb{C}(X), A) \to H^{i-1}(\mathbb{C}(D), A)$. This yields the alternative description

$$H^i_{\mathrm{nr}}(X, A) = \operatorname{Ker}\left(H^i(\mathbb{C}(X), A) \to \bigoplus_{D \subset X} H^{i-1}(\mathbb{C}(D), A)\right) .$$

More precisely, there exists a long exact sequence

$$\cdots \to H^i(F_1X, A) \to H^i(F_0X, A) \to \bigoplus_{D \subset X} H^{i-1}(F_0D, A) \to H^{i+1}(F_1X, A) \to \cdots ,$$

see [Sch23, Lemma 5.8] for more details and a generalization.

One can show that $H^i_{\mathrm{nr}}(X, A)$ is a stable birational invariant. This makes unramified cohomology a very useful tool for proving (stable) irrationality of algebraic varieties.

Since X is clearly a big open subset of itself, we have a map

$$H^i(X, A) \to H^i_{\mathrm{nr}}(X, A) .$$

This map is an isomorphism for $i = 1$ and surjective for $i = 2$ (by the above long exact sequence and by the fact that $H^i(F_jX, A) = H^i(X, A)$ for $j \geq \frac{i}{2}$, see [Sch23, Corollary 5.10]). For $i \geq 3$, however, this map is in general neither injective nor surjective.

The unramified cohomology groups $H^i_{\mathrm{nr}}(X, A)$ seem to become increasingly difficult to understand for increasing i. As we have just seen, $H^1_{\mathrm{nr}}(X, A) \cong H^1(X, A)$. For $A = \mathbb{Z}/m$, we further have $H^2_{\mathrm{nr}}(X, \mathbb{Z}/m) \cong \operatorname{Br}(X)[m]$, see e. g. [CT95, Proposition 4.2.3].

Colliot-Thélène and Voisin [CTV12] found the following surprising relation between the third unramified cohomology group $H^3_{\mathrm{nr}}(X, \mathbb{Z}/m)$ and the failure of the integral Hodge conjecture for X:

$$Z^4(X)[m] \cong \frac{H^3_{\mathrm{nr}}(X, \mathbb{Z}/m)}{H^3_{\mathrm{nr}}(X, \mathbb{Z}) \otimes \mathbb{Z}/m} \tag{5.1}$$

If $\mathrm{CH}_0(X)$ is supported on a surface, $H^3_{\mathrm{nr}}(X,\mathbb{Z})$ vanishes and we have $Z^4(X)[m] \cong H^3_{\mathrm{nr}}(X,\mathbb{Z}/m)$.

In [Sch23], Schreieder recently gave a more elementary proof of (5.1) and also obtained descriptions of $Z^{2k}(X)[m]$ for higher codimensions k in terms of refined unramified cohomology. As shown in [Sch23, Theorem 7.7], the image of the map $H^4(X,\mathbb{Z})[m] \to Z^4(X)[m]$ corresponds via the isomorphism (5.1) to the image of the map $H^3(X,\mathbb{Z}/m) \to H^3_{\mathrm{nr}}(X,\mathbb{Z}/m)$ introduced above. Hence, if the integral Hodge conjecture in codimension 2 on X is disproven via a non-zero class $\alpha \in H^3_{\mathrm{nr}}(X,\mathbb{Z}/m)$, the question whether this represents a counterexample of torsion type amounts to the question whether α, which is known to be liftable to a big open subset $U \subset X$, even lifts to all of X (but note that a global lift of α needs to agree with a given lift to a big open subset $U \subset X$ only on a non-empty open subset that is not necessarily big).

How can we explicitly describe classes in $H^i_{\mathrm{nr}}(X,\mathbb{Z}/m)$? We can construct elements in $H^i(\mathbb{C}(X),\mathbb{Z}/m)$ as an i-fold cup product of classes in $H^1(\mathbb{C}(X),\mathbb{Z}/m)$. By Hilbert's Theorem 90, the latter group is just $\mathbb{C}(X)^* \otimes \mathbb{Z}/m$. In fact, due to Voevodsky's proof [Voe03] of the Bloch–Kato conjecture, the map

$$(\mathbb{C}(X)^*)^{\otimes i} \otimes \mathbb{Z}/m \to H^i(\mathbb{C}(X),\mathbb{Z}/m)$$

descends to an isomorphism

$$K_i^M(\mathbb{C}(X)) \otimes \mathbb{Z}/m \xrightarrow{\cong} H^i(\mathbb{C}(X),\mathbb{Z}/m)\ .$$

Here, $K_\bullet^M(\mathbb{C}(X))$ denotes Milnor K-theory of the field $\mathbb{C}(X)$, i. e. the graded $\mathbb{Z}$-algebra generated by the symbols $\{a\}$ for $a \in \mathbb{C}(X)^*$ with the relations $\{ab\} = \{a\} + \{b\}$ for $a, b \in \mathbb{C}(X)^*$ and $\{a\} \cdot \{1-a\} = 0$ for $1 \neq a \in \mathbb{C}(X)^*$.

As we will see next, this algebraic description of $H^i(\mathbb{C}(X),\mathbb{Z}/m)$ has a nice geometric interpretation if we regard elements of $H^i(\mathbb{C}(X),\mathbb{Z}/m)$ as singular cohomology classes on a Zariski open subset $U \subset X$.

Let $n = \dim X$. Since X is a compact manifold of dimension $2n$, we have

$$H^i(X,A) \cong H_{2n-i}(X,A)$$

by Poincaré duality. Unfortunately, classical Poincaré duality does not hold for the Zariski open subsets $U \subsetneq X$ anymore, as they are not compact in the analytic topology. To accommodate this, we can replace singular homology by Borel–Moore homology $H_j^{\mathrm{BM}}(X,A)$, which agrees with $H_j(X,A)$ if X is compact. The Borel–Moore homology groups $H_j^{\mathrm{BM}}(X,A)$ are defined as the homology of the chain complex

$$C_0^{\mathrm{BM}}(X) \otimes A \to C_1^{\mathrm{BM}}(X) \otimes A \to C_2^{\mathrm{BM}}(X) \otimes A \to \cdots\ ,$$

where $C_j^{\mathrm{BM}}(X)$ differs from the classical $C_j(X)$ in singular homology only by allowing locally finite chains instead of just finite chains of j-simplices on X. Then we have

$$H^i(X, A) \cong H_{2n-i}^{\mathrm{BM}}(X, A)$$

even if X is not compact. However, it is still important here for X to be smooth.

Unlike in singular homology, pushforwards in Borel–Moore homology do not exist for arbitrary morphisms. However, similar to Chow groups, there exist pushforwards along proper morphisms and pullbacks along flat morphisms, see e. g. [Ful98, § 19.1].

Putting everything together, we have

$$H_{\mathrm{nr}}^i(X, A) \cong \operatorname{Im}\left(H_{2n-i}^{\mathrm{BM}}(F_1X, A) \to H_{2n-i}^{\mathrm{BM}}(F_0X, A)\right) ,$$

where $H_{2n-i}^{\mathrm{BM}}(F_jX, A)$ denotes the direct limit of $H_{2n-i}^{\mathrm{BM}}(U, A)$ over all Zariski open subsets $U \subset X$ such that $\operatorname{codim}_X(X \setminus U) > j$.

Using Poincaré duality, the cup product on singular cohomology induces an intersection pairing

$$\cap\colon H_i^{\mathrm{BM}}(X, A) \otimes H_j^{\mathrm{BM}}(X, A) \to H_{i+j-2n}^{\mathrm{BM}}(X, A) .$$

If the real submanifolds $M_1, M_2 \subset X$ intersect transversely, we have

$$[M_1] \cap [M_2] = [M_1 \cap M_2]$$

for the corresponding classes in Borel–Moore homology with $\mathbb{Z}/2$ coefficients (for general coefficients, one would need to take orientations into account). Even if the intersection is not transversal, $[M_1] \cap [M_2]$ can be represented by a cycle supported on $M_1 \cap M_2$.

A nice property of Borel–Moore homology is the existence of a long exact sequence

$$\cdots \to H_j^{\mathrm{BM}}(Z, A) \to H_j^{\mathrm{BM}}(X, A) \to H_j^{\mathrm{BM}}(U, A) \to H_{j-1}^{\mathrm{BM}}(Z, A) \to \cdots$$

for a Zariski open subset $U \subset X$ with complement $Z := X \setminus U$, see e. g. [Ful98, § 19.1]. Although the sequence

$$0 \to C_j^{\mathrm{BM}}(Z) \to C_j^{\mathrm{BM}}(X) \to C_j^{\mathrm{BM}}(U)$$

is in general not exact on the right, all Borel–Moore chains on a Zariski open subset $U \subset X$ appearing in this chapter can be represented by a chain on X. Such chain on X represents a cycle on U if and only if its boundary is supported on $X \setminus U$.

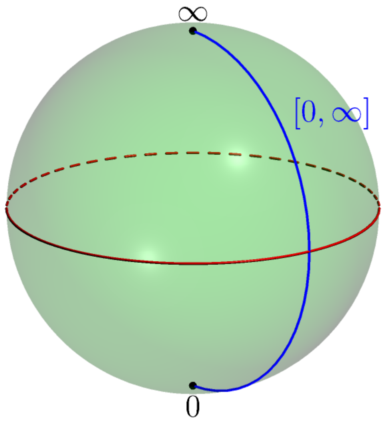

Figure 5.1.: Singular versus Borel–Moore homology of $\mathbb{P}^1 \setminus \{0, \infty\}$

Now we discuss a simple example of Borel–Moore homology, which is highly relevant for our later constructions. Let us consider $U = \mathbb{P}^1 \setminus \{0, \infty\}$, which is topologically a 2-punctured 2-sphere (see Figure 5.1). We have

$$H_1(U, \mathbb{Z}) \cong \mathbb{Z} \quad \text{and} \quad H_1^{\mathrm{BM}}(U, \mathbb{Z}) \cong \mathbb{Z} \ ,$$

but with entirely different generators: $H_1(U, \mathbb{Z})$ is generated by the class of the equator (red line in Figure 5.1), onto which U retracts. However, this class is trivial in Borel–Moore homology, as it is the boundary of each of the two punctured hemispheres. Instead, $H_1^{\mathrm{BM}}(U, \mathbb{Z})$ is generated by the class of any curve connecting 0 and ∞ (blue line in Figure 5.1). This does not give a well-defined class in singular homology because it cannot be represented by a finite chain of 1-simplices on U. After identifying the $\mathbb{C}$-points of $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$, we can explicitly represent this generator by the oriented interval $[0, \infty] \in C_1^{\mathrm{BM}}(\mathbb{P}^1)$, containing the positive real numbers, with boundary $\infty - 0$. This will play a key role in the geometric description of the norm residue map

$$K_i^M(\mathbb{C}(X)) \to \varinjlim_{U} H_{2n-i}^{\mathrm{BM}}(U, \mathbb{Z}) \ .$$

For simplicity, let us work with $\mathbb{Z}/2$ coefficients in the following, which will also be what we need later in section 5.4. Then we do not need to take orientations into account and can associate a chain in $C_j^{\mathrm{BM}}(X) \otimes \mathbb{Z}/2$ to any real submanifold (possibly with boundary) of X. However, by tracking orientations, it is possible to extend the subsequent statements in this section to integral coefficients.

Let $f \in \mathbb{C}(X)^*$ be a non-constant rational function. Equivalently, f is a dominant rational map $f \colon X \dashrightarrow \mathbb{P}^1$, or a flat morphism $f \colon U \to \mathbb{P}^1 \setminus \{0, \infty\}$ on a Zariski open subset $\emptyset \neq U \subset X$. Concretely, we can take U to be the complement of the zeros and poles of X. The unique generator of $H_1^{\mathrm{BM}}(\mathbb{P}^1 \setminus \{0,\infty\}, \mathbb{Z}/2) \cong \mathbb{Z}/2$ pulls back to a class $(f) \in H_{2n-1}^{\mathrm{BM}}(U, \mathbb{Z}/2)$. By functoriality of the isomorphism from Hilbert's Theorem 90, the image of (f) in $H_{2n-1}^{\mathrm{BM}}(F_0 X, \mathbb{Z}/2) \cong H^1(\mathbb{C}(X), \mathbb{Z}/2) \cong \mathbb{C}(X)^* \otimes \mathbb{Z}/2$ agrees with f itself.

For $f_1, \ldots, f_i \in \mathbb{C}(X)^*$, we write

$$(f_1, \ldots, f_i) := (f_1) \cap \cdots \cap (f_i) \in H_{2n-i}^{\mathrm{BM}}(U, \mathbb{Z}/2)$$

where $U \subset X$ is the complement of the zeros and poles of $f_1, \ldots, f_i$, i. e. we have $f_1, \ldots, f_i \in \mathcal{O}_X^*(U)$.

If we choose the chain $[0,\infty] \in C_1^{\mathrm{BM}}(\mathbb{P}^1) \otimes \mathbb{Z}/2$ from above as a representative for the generator of $H_1^{\mathrm{BM}}(\mathbb{P}^1 \setminus \{0,\infty\}, \mathbb{Z}/2)$, we therefore get a canonical chain

$$\mathcal{C}(f) := \overline{f^{-1}(0,\infty)} \in C_{2n-1}^{\mathrm{BM}}(X) \otimes \mathbb{Z}/2$$

representing (f), where the closure is taken in the analytic topology. Its boundary is supported on the zeros and poles of f.

In the special case where the zeros of f do not intersect the poles of f, the rational function f defines a flat morphism $f \colon X \to \mathbb{P}^1$ and we simply have

$$\mathcal{C}(f) = f^{-1}[0,\infty] \in C_{2n-1}^{\mathrm{BM}}(X) \otimes \mathbb{Z}/2\ .$$

In fact, $\mathcal{C}(f)$ always contains the zeros and poles of f. Hence, if we regard $f^{-1}(0)$ and $f^{-1}(\infty)$ by abuse of notation as the union of prime divisors appearing with positive/negative coefficient in the divisor associated to f, we also have $\mathcal{C}(f) = f^{-1}[0,\infty]$ in the general case, even if $f^{-1}(0)$ and $f^{-1}(\infty)$ are not disjoint.

Lemma 5.1. *Let $f, g \in \mathbb{C}(X)^*$ with $f + g = -1$. Then $\mathcal{C}(f) \cap \mathcal{C}(g)$ is supported on $f^{-1}(\infty) = g^{-1}(\infty)$. In particular, $(f) \cap (g) = 0 \in H_{2n-2}^{\mathrm{BM}}(U, \mathbb{Z}/2)$ where $\emptyset \neq U \subset X$ is chosen such that $f, g \in \mathcal{O}_X^*(U)$.*

Proof. Suppose we have $x \in \mathcal{C}(f) \cap \mathcal{C}(g)$ such that x is not a pole of f (and hence not of g). Then $f(x) \in [0,\infty)$ and $g(x) \in [0,\infty)$ are real numbers ≥ 0. This contradicts $f(x) + g(x) = -1$. □

This proves compatibility with the corresponding relation $\{f, g\} = 0$ in Milnor K-theory. Note that the usual definition of Milnor K-theory demands this relation for $f + g = 1$ instead (which does not really make a difference because $\{-1\} = 2 \cdot \{\sqrt{-1}\} = 0$ modulo 2). In this case, our chosen representatives $\mathcal{C}(f)$ and $\mathcal{C}(g)$ would not necessarily be disjoint on a Zariski open subset, and it would first require

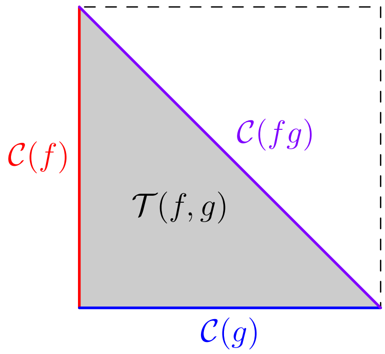

Figure 5.2.: Proof of Lemma 5.2

chains in $C_{2n}^{\mathrm{BM}}(X) \otimes \mathbb{Z}/2$ to "move" them before we can show $(f) \cap (g) = 0$. However, it turns out that during our computations in section 5.4 we are directly in the situation of Lemma 5.1.

To state the next lemma, let us introduce the complex argument

$$\arg\colon \mathbb{C}^* \to [0, 2\pi) \ .$$

Then $\mathcal{C}(f)$ is given by $\{\arg f = 0\}$ on the Zariski open subset where f is invertible.

Lemma 5.2. *Let $f, g \in \mathbb{C}(X)^*$. On the Zariski open subset $\emptyset \neq U \subset X$ where f and g are invertible, the chain $\mathcal{C}(f) + \mathcal{C}(g) - \mathcal{C}(fg)$ is the boundary of the real submanifold*

$$\mathcal{T}(f, g) := \{\arg f + \arg g \leq 2\pi\} \in C_{2n}^{\mathrm{BM}}(U) \otimes \mathbb{Z}/2 \ .$$

In particular, $(f) + (g) = (fg) \in H_{2n-1}^{\mathrm{BM}}(U, \mathbb{Z}/2)$.

Proof. By abuse of notation, let us pretend that $[0, 2\pi)$ has the quotient topology of $\mathbb{R}/2\pi\mathbb{Z}$ (this would be a more natural choice for the codomain of arg, but would not allow to write down the inequality defining $\mathcal{T}(f, g)$). Then the pair $(\arg f, \arg g)$ defines a continuous map from U to the torus $[0, 2\pi) \times [0, 2\pi)$. Clearly, the boundary of the triangle $\{x + y \leq 2\pi\}$ is given by the three lines $\{x = 0\}$, $\{y = 0\}$, and $\{x + y = 2\pi\}$, see Figure 5.2. Hence, the claim follows by pulling back to U, since $\arg(fg) = 0$ is equivalent to $\arg f + \arg g = 2\pi$ (up to the subset $\{\arg f = \arg g = 0\}$ of real codimension 2). □

This proves compatibility with the corresponding relation $\{fg\} = \{f\} + \{g\}$ in Milnor K-theory.

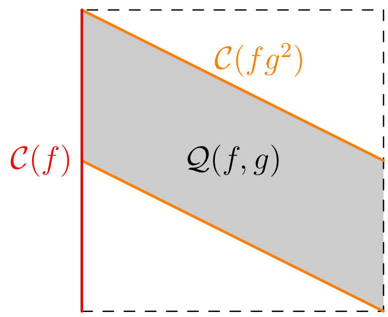

Figure 5.3.: Proof of Corollary 5.3

As a consequence of Lemma 5.1 and 5.2, there is a well-defined map

$$K_i^M(\mathbb{C}(X)) \otimes \mathbb{Z}/2 \to \varinjlim_U H_{2n-i}^{\mathrm{BM}}(U, \mathbb{Z}/2)$$

sending $\{f\}$ to the class of $\mathcal{C}(f)$. More precisely, the image of a symbol $\{f_1, \dots, f_i\}$ lies in $H_{2n-i}^{\mathrm{BM}}(U, \mathbb{Z}/2)$, where $U \subset X$ is the complement of the zeros and poles of $f_1, \dots, f_i$.

For our computation in section 5.4, we will use the explicit descriptions of chains like $\mathcal{T}(f,g)$ that correspond to relations in Milnor K-theory between the classes of our chosen system of canonical representatives. The following consequence of Lemma 5.2 will appear repeatedly:

Corollary 5.3. *Let $f, g \in \mathbb{C}(X)^*$. On the Zariski open subset $\emptyset \neq U \subset X$ where f and g are invertible, the chain $\mathcal{C}(f) - \mathcal{C}(fg^2)$ is the boundary of the real submanifold*

$$\mathcal{Q}(f,g) := \left\{\pi - \tfrac{1}{2}\arg f \leq \arg g \leq 2\pi - \tfrac{1}{2}\arg f\right\} \in C_{2n}^{\mathrm{BM}}(U) \otimes \mathbb{Z}/2\,.$$

In particular, $(f) = (fg^2) \in H_{2n-1}^{\mathrm{BM}}(U, \mathbb{Z}/2)$.

Proof. By Lemma 5.2, $\mathcal{C}(fg^2) - \mathcal{C}(f)$ is the boundary of $\mathcal{T}(f,g) + \mathcal{T}(fg,g)$. After a subdivision into smaller triangles, this sum turns out to be equal to $\mathcal{Q}(f,g)$ modulo 2. Alternatively, one could directly compute the boundary of $\mathcal{Q}(f,g)$ to obtain the result, see Figure 5.3. $\square$

5.3. A smooth resolution of Schreieder's conic bundle

In the proof of [Sch19, Theorem 1.5] for $N = 4$, Schreieder considers (up to birational equivalence) the unirational conic bundle $f \colon Y \to \mathbb{P}^3$ given by

$$Y = \{x_0 x_1 g \cdot y_0^2 + x_2 x_3 \cdot y_1^2 + y_2^2 = 0\} \subset \mathbb{P}(\mathcal{O}(-3) \oplus \mathcal{O}(-1) \oplus \mathcal{O})$$

and

$$g = t^2 G^2 - x_0 x_1 x_2 x_3 \ ,$$

where $t \in \mathbb{C}$ is transcendental and $G \in \mathbb{Q}[x_0, x_1, x_2, x_3]$ is a homogeneous polynomial of degree 2 such that all monomials x_i^2 appear with a non-zero coefficient. Throughout this chapter, we will set

$$G = x_0^2 + x_1^2 + x_2^2 + x_3^2 \ .$$

See also Remark 5.6 regarding the choice of G.

Let X be a smooth projective variety birational to Y. Let

$$\alpha := \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in H_3^{\mathrm{BM}}(U, \mathbb{Z}/2)$$

where $U = \{x_0 x_1 x_2 x_3 \neq 0\} \subset \mathbb{P}^3$. We can represent α by the transversal intersection

$$\mathcal{C}\left(\frac{x_1}{x_0}\right) \cap \mathcal{C}\left(\frac{x_2}{x_0}\right) \cap \mathcal{C}\left(\frac{x_3}{x_0}\right) = \left\{\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \in (0, \infty)\right\} \subset U(\mathbb{R})$$

on U. Note that the real manifold $U(\mathbb{R})$ has 8 connected components, and this 3-cycle is one of them. Its closure is a solid tetrahedron and consists precisely of the points in $\mathbb{P}^3(\mathbb{R})$ given by four non-negative homogeneous coordinates. This closure is an element of $C_3^{\mathrm{BM}}(\mathbb{P}^3) \otimes \mathbb{Z}/2$ and will be our canonical representative for α. Its boundary is the sum of four triangles, each supported on one of the divisors $\{x_i = 0\}$.

As shown in [Sch19, Proposition 5.1], $f^*\alpha$ is an unramified cohomology class, i. e. inside the direct limit $H_3(\mathbb{C}(Y), \mathbb{Z}/2) \cong H_5^{\mathrm{BM}}(\mathbb{C}(X), \mathbb{Z}/2)$ it can be replaced by a class in $H_5^{\mathrm{BM}}(U, \mathbb{Z}/2)$ for a big open subset $U \subset X$. At the same time, this class is non-zero in $H_3(\mathbb{C}(Y), \mathbb{Z}/2)$ by [Sch19, Proposition 6.1]. Hence, the integral Hodge conjecture for X fails in codimension 2 due to (5.1). As seen in section 5.2, our goal is to decide whether $f^*\alpha$ is induced by a class in $H_5^{\mathrm{BM}}(X, \mathbb{Z}/2)$ for some smooth projective model X of Y.

In order to obtain a smooth projective model X, we want to resolve the singularities of Y by blowing up $\mathbb{P}^3$ along components of the degeneracy divisor until we arrive at a smooth projective variety S birational to $\mathbb{P}^3$ that admits a smooth conic bundle $X \to S$ birational to Y.

The following smoothness criterion roughly says that smoothness is guaranteed once the degeneracy divisor has simple normal crossings, each two intersecting components "belong to different y_i", and there are no triple intersections:

Lemma 5.4. *Let S be a smooth projective variety. For $i \in \{0, 1, 2\}$, let $\mathcal{L}_i$ be a line bundle on S and let a_i be a global section of $(\mathcal{L}_i^\vee)^{\otimes 2}$. Then*

$$X = \{a_0 \cdot y_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 = 0\} \subset \mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$$

defines a smooth conic bundle over S, provided that the following conditions are satisfied:

(i) *each $D_i := \{a_i = 0\} \subset X$ is smooth (but not necessarily irreducible),*

(ii) *$D_0 \cup D_1 \cup D_2 = \{a_0a_1a_2 = 0\} \subset X$ is a (reduced) simple normal crossing divisor,*

(iii) *$D_0 \cap D_1 \cap D_2 = \emptyset$.*

Proof. By (iii), the morphism $X \to S$ is flat, and each fibre is a conic in $\mathbb{P}^2$. It remains to verify that X is smooth. This can be checked étale-locally. By (ii), there exist local coordinates $x_1, \ldots, x_n$ of S such that each a_i is locally given by the product of some subset of $\{x_1, \ldots, x_n\}$, and these three subsets are pairwise disjoint. We may also assume that $\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ is trivial over these local coordinates. In light of (i) and (iii), up to renumbering we are left with the following possibilities for a local equation of X inside $\mathbb{A}^n \times \mathbb{P}^2$:

- $y_0^2 + y_1^2 + y_2^2 = 0$
- $x_1y_0^2 + y_1^2 + y_2^2 = 0$
- $x_1y_0^2 + x_2y_1^2 + y_2^2 = 0$

In each case, one can easily verify smoothness via the Jacobian criterion. □

By [Sar82, Theorem 1.13], it is possible to find a model X satisfying the conditions of Lemma 5.4. However, since we will later work with Borel–Moore homology classes on X, we want to have more control over the resolution. Therefore, instead of refering to the general results from [Sar82], we will explicitly construct a smooth model X with suitable properties.

To this end, we additionally consider the conic bundle $f'\colon Y' \to \mathbb{P}^3$ given by

$$Y' = \{x_0x_1 \cdot y_0^2 + x_2x_3 \cdot y_1^2 + y_2^2 = 0\} \subset \mathbb{P}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}) ,$$

which differs from Y only by omitting the factor g. This bundle can be easily resolved by the blow-up $\phi'\colon S' \to \mathbb{P}^3$ along the two disjoint lines $\{x_0 = x_1 = 0\}$ and $\{x_2 = x_3 = 0\}$. This results in a smooth conic bundle $\tilde{f}'\colon X' \to S'$ and a birational equivalence $X' \overset{\sim}{\dashrightarrow} Y'$ such that the diagram of rational maps

$$\begin{tikzcd} X' \arrow[r, dashed, "\sim"] \arrow[d, "\tilde{f}'"'] & Y' \arrow[d, "f"] \\ S' \arrow[r, "\phi'"] & \mathbb{P}^3 \end{tikzcd}$$

commutes. Let $Z' := \phi'^{-1}(\{x_0x_1x_2x_3 = 0\})$ be the inverse image of the degeneracy locus of Y'. Clearly, Z' consists of the strict transforms of the divisors $\{x_i = 0\}$ together with the two exceptional divisors over $\{x_0 = x_1 = 0\}$ and $\{x_2 = x_3 = 0\}$.

Note that Y and Y' are isomorphic over the divisor $\{x_0x_1x_2x_3 = 0\}$ on which the boundary of α is supported. Our aim is to construct a resolution $X \to S$ such that the inverse image of $\{x_0x_1x_2x_3 = 0\}$ in X contains a closed subset Z which is, up to blow-ups of certain points, isomorphic to Z'. Additionally, we want that the boundaries of the respective pullbacks of α agree under this isomorphism. See Proposition 5.8 for the precise statement.

In order to achieve this, a crucial observation is that the component $\{g = 0\}$ of the degeneracy divisor of Y does not meet α:

Lemma 5.5. *If $t \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$, then $\{g = 0\}$ is disjoint from our canonical representative $\alpha \in C_3^{\mathrm{BM}}(\mathbb{P}^3) \otimes \mathbb{Z}/2$.*

Proof. As we have seen, α is supported on $\mathbb{P}^3(\mathbb{R})$. Suppose that $[x_0 : x_1 : x_2 : x_3]$ with $x_0, x_1, x_2, x_3 \in \mathbb{R}$ is an $\mathbb{R}$-point of $\{g = 0\}$. Then we have $x_0^2 + x_1^2 + x_2^2 + x_3^2 > 0$ and thus

$$t^2 = \frac{x_0x_1x_2x_3}{(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2} \in \mathbb{R}\ ,$$

a contradiction. $\square$

Since $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ is uncountable, the additional assumption on the transcendental number $t \in \mathbb{C}$ is always satisfied up to a suitable automorphism of $\mathbb{C}$ fixing $\overline{\mathbb{Q}}$.

Remark 5.6. The proof of Lemma 5.5 generalizes to all positive definite quadratic forms G.

Remark 5.7. The statement in Lemma 5.5 is true for certain $t \in \mathbb{R}$ as well. Indeed, for $t > \frac{1}{4}$ we obtain

$$t^2(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2 > x_0x_1x_2x_3$$

for all $(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 \setminus \{0\}$ by the inequality between arithmetic and geometric mean.

Proposition 5.8. *There exists a morphism $\phi\colon S \to \mathbb{P}^3$, which is a composition of blow-ups along smooth centres, and a smooth conic bundle $\tilde{f}\colon X \to S$ together with a birational equivalence $X \overset{\sim}{\dashrightarrow} Y$ such that the following conditions are satisfied:*

(1) *The diagram of rational maps*

$$\begin{tikzcd} X \arrow[r, dashed, "\sim"] \arrow[d, "\tilde{f}"'] & Y \arrow[d, "f"] \\ S \arrow[r, "\phi"] & \mathbb{P}^3 \end{tikzcd}$$

commutes.

(2) *There exists a closed subscheme*

$$Z \subset \phi^{-1}(\{x_0x_1x_2x_3 = 0\})$$

and a morphism $Z \to Z'$ such that the conic bundle X restricted to Z is the pullback of the conic bundle X' restricted to Z'. The morphism $Z \to Z'$ corresponds to finitely many iterated blow-ups of the strict transforms of the divisors $\{x_i = 0\}$ in points lying on the strict transforms of the lines $\{x_i = x_j = 0\}$ for $ij \notin \{01, 23\}$ or on the exceptional divisors of previous such blow-ups.

(3) *The class*

$$\alpha \in H^3(\mathbb{C}(\mathbb{P}^3), \mathbb{Z}/2) \cong H^3(\mathbb{C}(S), \mathbb{Z}/2)$$

on S can be represented by a chain in $C_3^{\mathrm{BM}}(S) \otimes \mathbb{Z}/2$ whose boundary is supported on Z and does not meet the exceptional divisor of $Z \to Z'$. This boundary agrees with the boundary of the canonical representative $\alpha \in C_3^{\mathrm{BM}}(S') \otimes \mathbb{Z}/2$.

As a preparation for the proof of this proposition, let us first show the following general lemma which allows to "absorb" any square inside a coefficient of a diagonal conic bundle into the conic bundle itself.

Lemma 5.9. *Let*

$$X = \{a_0 \cdot y_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 = 0\} \subset \mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2)$$

be a conic bundle as in Lemma 5.4, not necessarily smooth. Suppose that the divisor $D_0 = \{a_0 = 0\}$ has multiplicity ≥ 2 along a prime divisor $E \subset S$. Then X is birational over S to a conic bundle where the multiplicity of D_0 along E is reduced by 2, and all other multiplicities remain unchanged.

Proof. Let b be the global section of $\mathcal{L} := \mathcal{O}_S(E)$ corresponding to E. Then we may consider the conic bundle $\tilde{X}$ over S given by

$$\tilde{X} = \left\{ \frac{a_0}{b^2} \cdot \tilde{y}_0^2 + a_1 \cdot y_1^2 + a_2 \cdot y_2^2 = 0 \right\} \subset \mathbb{P}((\mathcal{L}_0 \otimes \mathcal{L}) \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) .$$

Note that $\frac{a_0}{b^2}$ defines a global section of $((\mathcal{L}_0 \otimes \mathcal{L})^\vee)^{\otimes 2} = (\mathcal{L}_0^\vee)^{\otimes 2} \otimes (\mathcal{L}^\vee)^{\otimes 2}$ because a_0 vanishes with multiplicity at least 2 along $\{b = 0\}$. We observe that $\tilde{X}$ has exactly the stated multiplicities. Finally, one can check that $\tilde{y}_0 \mapsto b \cdot y_0$ induces a birational equivalence $X \dashrightarrow \tilde{X}$ over S. $\square$

Proof of Proposition 5.8. Let $H_i := \{x_i = 0\}$ and $D := \{g = 0\}$ be the components of the degeneracy divisor of $f\colon Y \to \mathbb{P}^3$. To simplify the notation, we denote their strict transforms after each step still by the same letters. Note that the quartic surface D is tangent to each of the four transversally intersecting planes H_0, H_1, H_2, H_3.

We first blow up the disjoint lines $H_0 \cap H_1$ and $H_2 \cap H_3$. This introduces two exceptional divisors E_{01} and E_{23}. Pulling back the conic bundle Y to this blowup, we see that the coefficient in front of y_0^2 vanishes with multiplicity 2 along E_{01}. Similarly, the coefficient in front of y_1^2 vanishes with multiplicity 2 along E_{23}. Hence, by Lemma 5.9 we can erase E_{01} and E_{23} from the degeneracy divisor through a birational modification of the conic bundle. From now on, we have $H_0 \cap H_1 = \emptyset$ and $H_2 \cap H_3 = \emptyset$ for the strict transforms H_i.

Next we blow up the reduced subscheme associated to $H_0 \cap D$, i. e. the conic $\{x_0 = G = 0\}$ (recall that D denotes the strict transform of our original divisor D on $\mathbb{P}^3$). This introduces an exceptional divisor E_0. Again E_0 appears in the y_0^2-part of the degeneracy divisor of the pullback bundle with multiplicity 2, and can thus be eliminated from the degeneracy divisor by a suitable birational transformation due to Lemma 5.9. Now the divisors H_0 and D intersect transversely along a smooth curve. By blowing up $H_0 \cap D$, we get another exceptional divisor E_0', which can be excluded from the degeneracy divisor by the same argument as before. The strict transforms H_0 and D are now disjoint.

The analogous pair of blow-ups is then applied to $H_1 \cap D$, and after that to $H_2 \cap D$ and $H_3 \cap D$, resulting in 6 further blow-ups.

While the discussion for $H_1 \cap D$ is the same, the situation is slightly more complicated for $H_2 \cap D$ and $H_3 \cap D$. Indeed, the parts of the degeneracy divisor corresponding to y_0^2 and y_1^2 will both contain the exceptional divisor E_2 with multiplicity 1 each. However, before the second blow-up of $H_2 \cap D$, we are in the situation that also H_2 and E_2 as well as D and E_2 intersect transversely along the smooth curve $H_2 \cap D$. Therefore, the blow-up of $H_0 \cap D$ causes the coefficients in front of both y_0^2 and y_1^2 to vanish with multiplicity 2 along E_2'. Thus, we may again perform a birational modification such that E_2' is not part of the degeneracy divisor anymore. Although E_2 is still present in the degeneracy divisor, we may achieve by another birational transformation that E_2 appears with multiplicity 1 in the divisor associated to the coefficient of y_2^2, instead of y_0^2 and y_1^2. Since E_2 does not intersect any of the considered divisors anymore (except E_2'), it does not cause any issues regarding the requirements of Lemma 5.4. The same argument applies to E_3 and E_3'.

At this point, we have a birational model of Y with degeneracy divisor $H_0 + H_1 + H_2 + H_3 + D + E_2 + E_3$ satisfying all conditions of Lemma 5.4 except possibly that

D might still be singular. Since D is already disjoint from all other components of the degeneracy locus, we can use the general algorithm from [Sar82] to blow up subvarieties of D (or of the exceptional divisors introduced during this last step) until we arrive at a smooth conic bundle $\tilde{f}\colon X \to S$.

Condition (1) is satisfied by construction. Condition (2) is also satisfied by taking Z to be the union of the H_i with E_{01} and E_{23}. Note that H_i is blown up in two distinct non-real points when blowing up $H_j \cap D$ for $ij \notin \{01, 23\}$. Finally, by Lemma 5.5 the class α is only affected by the first two blow-ups. These blow-ups are precisely $\phi'\colon S' \to \mathbb{P}^3$. This shows condition (3). $\square$

5.4. Geometric study of the unramified cohomology class

Recall that S' was defined as the blow-up of $\mathbb{P}^3$ along the two disjoint lines $\{x_0 = x_1 = 0\}$ and $\{x_2 = x_3 = 0\}$. Hence, we have $S' = \operatorname{Proj} \mathbb{C}[\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t_{01}, t_{23}]$ with respect to the following trigrading:

$$\begin{aligned} \deg \tilde{x}_0 = \deg \tilde{x}_1 &= (1,0,0) \qquad & \deg t_{01} &= (-1,0,1) \\ \deg \tilde{x}_2 = \deg \tilde{x}_3 &= (0,1,0) \qquad & \deg t_{23} &= (0,-1,1) \end{aligned}$$

The blow-down map $S' \to \mathbb{P}^3$ corresponds to $x_0 \mapsto t_{01}\tilde{x}_0$, $x_1 \mapsto t_{01}\tilde{x}_1$, $x_2 \mapsto t_{23}\tilde{x}_2$, and $x_3 \mapsto t_{23}\tilde{x}_3$. From this description, we can see that each of the four divisors in S' cut out by one $\tilde{x}_i$ is isomorphic to the Hirzebruch surface Σ_1 (the total space of the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$), while the two divisors in S' cut out by t_{01} and t_{23}, respectively, are isomorphic to the Hirzebruch surface $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In total, the six divisors intersect like the six sides of a cube (i. e. their intersection graph is an octahedron), each of these intersections is a $\mathbb{P}^1$, and each non-empty triple intersection is a single point, see also Figure 5.4.

The conic bundle $X' \to S'$ can thus be regarded as the hypersurface of degree $(0,0,0,2)$ cut out by

$$\tilde{x}_0\tilde{x}_1 y_0^2 + \tilde{x}_2\tilde{x}_3 y_1^2 + y_2^2 = 0$$

inside the weighted projective space

$$\mathbb{P}(\mathcal{O}_{S'}(-1,0,0) \oplus \mathcal{O}_{S'}(0,-1,0) \oplus \mathcal{O}_{S'}) = \operatorname{Proj} \mathbb{C}[\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, t_{01}, t_{23}, y_0, y_1, y_2]$$

with the following multigrading:

$$\begin{aligned} \deg \tilde{x}_0 = \deg \tilde{x}_1 &= (1,0,0,0) \qquad & \deg t_{01} &= (-1,0,1,0) \\ \deg \tilde{x}_2 = \deg \tilde{x}_3 &= (0,1,0,0) \qquad & \deg t_{23} &= (0,-1,1,0) \end{aligned}$$
$$\deg y_0 = (-1,0,0,1) \quad \deg y_1 = (0,-1,0,1) \quad \deg y_2 = (0,0,0,1)$$

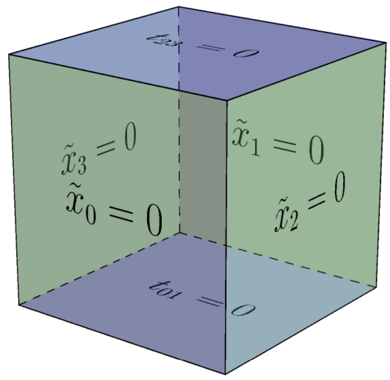

Figure 5.4.: The six divisors on the blow-up S'

The class $\alpha \in H_3^{\mathrm{BM}}(\mathbb{C}(\mathbb{P}^3), \mathbb{Z}/2) = H_3^{\mathrm{BM}}(\mathbb{C}(S'), \mathbb{Z}/2)$, which was represented on $\mathbb{P}^3$ by the interior of a tetrahedron, can now be represented on S' by the interior of the cube bounded by the aforementioned six divisors in the real points of S'. In equations, we have

$$\alpha = \left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 t_{23}}{\tilde{x}_0 t_{01}} \right) .$$

As a chain on S', the boundary of α is the sum of six squares, each of them supported on one of the six Hirzebruch surfaces.

Let $W = \tilde{f}^{-1} Z$ and $W' = \tilde{f}'^{-1} Z'$. In particular, there is a birational morphism $W \to W'$ by condition (2) of Proposition 5.8. Let

$$\beta = \beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_{01} + \beta_{23}$$

be the boundary of the canonical representative $\alpha \in C_3^{\mathrm{BM}}(S') \otimes \mathbb{Z}/2$, where

$$\beta_i \in C_2^{\mathrm{BM}}(\{\tilde{x}_i = 0\}) \otimes \mathbb{Z}/2 \quad \text{and} \quad \beta_{ij} \in C_2^{\mathrm{BM}}(\{t_{ij} = 0\}) \otimes \mathbb{Z}/2 .$$

Let $F_i' \subset Z'$ be the fibre of the Hirzebruch surface $\{\tilde{x}_i = 0\} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ at $[1 : -1] \in \mathbb{P}^1$. In other words, we have e. g. $F_0' = \{\tilde{x}_0 = \tilde{x}_2 + \tilde{x}_3 = 0\}$. Note that F_i' is disjoint from β_i.

On F_0', using $\tilde{x}_2 + \tilde{x}_3 = 0$ the conic bundle is given by

$$\tilde{x}_2 \tilde{x}_3 y_1^2 + y_2^2 = 0 \iff (\tilde{x}_2 y_1 + y_2)(\tilde{x}_3 y_1 + y_2) = 0 .$$

Let $K_0' := \{\tilde{x}_2 y_1 + y_2\} \subset f'^{-1} F_0'$. Similarly, we define the components

$$K_1' := \{\tilde{x}_2 y_1 + y_2\} \subset f'^{-1} F_1' ,$$
$$K_2' := \{\tilde{x}_0 y_0 + y_2\} \subset f'^{-1} F_2' ,$$
$$K_3' := \{\tilde{x}_0 y_0 + y_2\} \subset f'^{-1} F_3' .$$

Let $\kappa \in C_4^{\mathrm{BM}}(W') \otimes \mathbb{Z}/2$ be the cycle $K'_0 + K'_1 + K'_2 + K'_3$.

Since F'_i is disjoint from the lines $\{\tilde{x}_i = \tilde{x}_j = 0\}$, we observe that $W \to W'$ is an isomorphism above each F'_i. Thus we obtain subvarieties $K_i \subset W$ corresponding to $K'_i \subset W'$.

Our goal is to prove:

Proposition 5.10. *The cycle $f'^*\beta \in C_4^{\mathrm{BM}}(W') \otimes \mathbb{Z}/2$ can be written as*

$$f'^*\beta = \partial\gamma + \kappa$$

for some chain $\gamma \in C_5^{\mathrm{BM}}(W') \otimes \mathbb{Z}/2$. Moreover, the support of γ is contained in the open subset where $W \to W'$ is an isomorphism.

This will imply the following:

Corollary 5.11. *Twice Schreieder's non-algebraic class in $H^4(X, \mathbb{Z})$ can be represented by the algebraic cycle $K_0 + K_1 + K_2 + K_3$ on X.*

Hence, the question whether Schreider's example is 2-torsion leads to the question whether $\kappa \in H^4(X, \mathbb{Z}/2)$ is trivial. Note that this is indeed the case on X', since the conic bundle above $\{\tilde{x}_i = \tilde{x}_j = 0\}$ (with $ij \notin \{01, 23\}$) is just a projective line with multiplicity 2, and both K'_i and K'_j are homologous to this $\mathbb{P}^1$-bundle in $\{\tilde{x}_i = 0\}$ and $\{\tilde{x}_j = 0\}$, respectively. On X, however, the surfaces K_i and K_j on adjacent sides of our cube differ in homology exactly by the reduced bundle above the exceptional divisors arising from the point blow-ups on $\{\tilde{x}_i = \tilde{x}_j = 0\}$. The homology at these exceptional divisors would thus indicate whether Schreieder's counterexample to the integral Hodge conjecture is of torsion type.

Proof of Corollary 5.11. By property (2) of Proposition 5.8 and the condition on the support of γ in Proposition 5.10, we may also view γ as a chain on X. The same applies to the cycle κ. Since γ is supported on the subvariety $W \subset X$ of codimension 1, we may replace $f^*\alpha$ by $f^*\alpha + \gamma$ in the direct limit $H_5^{\mathrm{BM}}(\mathbb{C}(X), \mathbb{Z}/2)$. By property (3) of Proposition 5.8, the boundary of $f^*\alpha$, viewed in $C_4^{\mathrm{BM}}(W) \otimes \mathbb{Z}/2$, corresponds on W' to $f'^*\beta = \partial\gamma + \kappa \in C_4^{\mathrm{BM}}(W') \otimes \mathbb{Z}/2$. Therefore, we have $\partial(f^*\alpha + \gamma) = \kappa$ on X. Since κ is supported in codimension 2, it follows that $f^*\alpha + \gamma$ (and thus $f^*\alpha$) is unramified. As the proof of [Sch23, Theorem 7.7] shows, the corresponding element in $Z^4(X)[2]$ can be represented by μ, where $\mu \in C_4^{\mathrm{BM}}(X)$ is chosen such that $\partial(f^*\alpha + \gamma) = \kappa + 2\mu$ in $C_4^{\mathrm{BM}}(X)$ (instead of $C_4^{\mathrm{BM}}(X) \otimes \mathbb{Z}/2$). Therefore, 2μ can be represented by κ. $\square$

Proof of Proposition 5.10. Let us first see why $f'^*\beta$ vanishes in $H_4^{\mathrm{BM}}(F_0W',\mathbb{Z}/2)$, i. e. $f'^*\alpha$ is unramified. For $i \in \{0,1\}$, using Corollary 5.3 we have

$$\begin{aligned}
[f'^*\beta_i] &= \left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 t_{23}}{t_{01}}\right) = \left(\frac{\tilde{x}_3}{\tilde{x}_2}\left(\frac{\tilde{x}_2 y_1}{y_2}\right)^2, \frac{\tilde{x}_2 t_{23}}{t_{01}}\right) \\
&= 0 \in H_4^{\mathrm{BM}}(F_0\{\tilde{x}_i = 0\}, \mathbb{Z}/2)
\end{aligned} \tag{5.2}$$

because

$$\frac{\tilde{x}_3}{\tilde{x}_2}\left(\frac{\tilde{x}_2 y_1}{y_2}\right)^2 = \tilde{x}_2\tilde{x}_3\left(\frac{y_1}{y_2}\right)^2 = -1\ .$$

Similarly, for $i \in \{2,3\}$ we have

$$\begin{aligned}
[f'^*\beta_i] &= \left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{t_{23}}{\tilde{x}_0 t_{01}}\right) = \left(\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2, \frac{t_{23}}{\tilde{x}_0 t_{01}}\right) \\
&= 0 \in H_4^{\mathrm{BM}}(F_0\{\tilde{x}_i = 0\}, \mathbb{Z}/2)
\end{aligned} \tag{5.3}$$

because

$$\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2 = \tilde{x}_0\tilde{x}_1\left(\frac{y_0}{y_2}\right)^2 = -1\ .$$

Finally, for $ij \in \{01,23\}$ we have

$$\begin{aligned}
[f'^*\beta_{ij}] &= \left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_3}{\tilde{x}_2}\right) = \left(\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2, \frac{\tilde{x}_3}{\tilde{x}_2}\right) \\
&= \left(\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2, \frac{\tilde{x}_3}{\tilde{x}_2}\left(\frac{\tilde{x}_2 y_1}{y_2}\right)^2\right) \\
&= 0 \in H_4^{\mathrm{BM}}(F_0\{t_{ij} = 0\}, \mathbb{Z}/2)
\end{aligned} \tag{5.4}$$

because

$$\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2 + \frac{\tilde{x}_3}{\tilde{x}_2}\left(\frac{\tilde{x}_2 y_1}{y_2}\right)^2 = \tilde{x}_0\tilde{x}_1\left(\frac{y_0}{y_2}\right)^2 + \tilde{x}_2\tilde{x}_3\left(\frac{y_1}{y_2}\right)^2 = -1\ ,$$

so Lemma 5.1 applies.

Due to the preceding computations in homology, on each of the six divisors (or rather their pullbacks via f') there exists a 5-chain whose boundary agrees with the corresponding part of $f'^*\beta$ on the Zariski open subset where all appearing rational functions are invertible, i. e. when we remove the four other divisors intersecting the considered one. In order to obtain a 5-chain on W' whose boundary agrees with $f'^*\beta$ up to the additional components K'_i, we need to check that these 5-chains glue together along the $\mathbb{P}^1$ intersections of our divisors (or rather their pullbacks via f'), i. e. along the 12 edges of the cube.

In the following, we will check this in detail for the three edges $\{\tilde{x}_0 = t_{01} = 0\}$, $\{\tilde{x}_2 = t_{01} = 0\}$, and $\{x_0 = x_2 = 0\}$. All other edges behave analogously to one of these three edges.

We first need to discuss more precisely how the real square $\beta_0 \in C_2^{\mathrm{BM}}(\{\tilde{x}_0 = 0\}) \otimes \mathbb{Z}/2$ (and analogously $\beta_2 \in C_2^{\mathrm{BM}}(\{\tilde{x}_2 = 0\}) \otimes \mathbb{Z}/2$) can be described. In homology, we have

$$[\beta_0] = \left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 t_{23}}{t_{01}}\right) = \left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_3 t_{23}}{t_{01}}\right) .$$

However, the 2-chains

$$\mathcal{C}\left(\frac{\tilde{x}_3}{\tilde{x}_2}\right) \cap \mathcal{C}\left(\frac{\tilde{x}_2 t_{23}}{t_{01}}\right)$$

and

$$\mathcal{C}\left(\frac{\tilde{x}_3}{\tilde{x}_2}\right) \cap \mathcal{C}\left(\frac{\tilde{x}_3 t_{23}}{t_{01}}\right)$$

describe β_0 correctly only on the Zariski open subsets $\{\tilde{x}_2 \neq 0\}$ or $\{\tilde{x}_3 \neq 0\}$, respectively. These 2-chains fully contain the divisors $\{\tilde{x}_2 = 0\}$ or $\{\tilde{x}_3 = 0\}$, respectively, and are thus not supported on the $\mathbb{R}$-points of S'. As we have seen in section 5.2, β_0 is the closure (in the analytic topology) of any of these two 2-chains on $\{\tilde{x}_2\tilde{x}_3 \neq 0\}$. This is different from both 2-chains because taking closures does not commute with taking intersections.

We claim that

$$\beta_0 = \mathcal{C}\left(\frac{\tilde{x}_3}{\tilde{x}_2}\right) \cap \mathcal{C}\left(\frac{(\tilde{x}_2 + \tilde{x}_3) t_{23}}{t_{01}}\right) .$$

Indeed, for a point in $\mathcal{C}\left(\tilde{x}_3/\tilde{x}_2\right)$ we can choose homogeneous coordinates with $\tilde{x}_2, \tilde{x}_3 \in [0, \infty)$ and thus $\tilde{x}_2 + \tilde{x}_3 > 0$ (as $\tilde{x}_2 = \tilde{x}_3 = 0$ is impossible), so $\mathcal{C}\left((\tilde{x}_2 + \tilde{x}_3) t_{23}/t_{01}\right)$ agrees with $\mathcal{C}\left(\tilde{x}_2 t_{23}/t_{01}\right)$ and $\mathcal{C}\left(\tilde{x}_3 t_{23}/t_{01}\right)$ on $\{\tilde{x}_2 \neq 0\}$ and $\{\tilde{x}_3 \neq 0\}$, respectively.

As our previous computation (5.2) in homology shows, $f'^*\beta_0 \in C_4^{\mathrm{BM}}(\{\tilde{x}_0 = 0\}) \otimes \mathbb{Z}/2$ is due to Corollary 5.3 the boundary of

$$\gamma_0 := \mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right) \cap \mathcal{C}\left(\frac{(\tilde{x}_2 + \tilde{x}_3) t_{23}}{t_{01}}\right) .$$

A priori, γ_0 is only defined on the Zariski open subset where the appearing rational functions are invertible. By abuse of notation, we identify γ_0 with its closure (in the analytic topology) to obtain a 5-chain $\gamma_0 \in C_5^{\mathrm{BM}}(\{\tilde{x}_0 = 0\}) \otimes \mathbb{Z}/2$. This might add further summands to the boundary of γ_0. These 4-chains (except for $f'^*\beta_0$) are supported on the divisors $\{\tilde{x}_2 = 0\}$, $\{\tilde{x}_3 = 0\}$, $\{\tilde{x}_2 + \tilde{x}_3 = 0\}$, $\{y_1 = 0\}$, $\{y_2 = 0\}$, $\{t_{01} = 0\}$, and $\{t_{23} = 0\}$ of our considered threefold $\{\tilde{x}_0 = 0\} \subset X'$.

Since the boundary of $\mathcal{Q}(u, v)$ contains the divisor $\{v = 0\}$ with even multiplicity, the boundary of γ_0 along $\{\tilde{x}_0 = y_1 = 0\}$ and $\{\tilde{x}_0 = y_2 = 0\}$ vanishes. On $\{\tilde{x}_0 = t_{01} = 0\}$

and $\{\tilde{x}_0 = t_{23} = 0\}$, we obtain the 4-chain

$$\mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right) .$$

On $\{\tilde{x}_0 = \tilde{x}_2 = 0\}$ and $\{\tilde{x}_0 = \tilde{x}_3 = 0\}$, the restriction of γ_0 is given by

$$\mathcal{C}\left(\frac{t_{23}}{t_{01}}\right) .$$

On $\{\tilde{x}_0 = \tilde{x}_2 + \tilde{x}_3 = 0\}$, we get

$$\mathcal{Q}\left(-1, \frac{\tilde{x}_2 y_1}{y_2}\right) .$$

As we can see from the definition of $\mathcal{Q}$, this selects from the two components $\{\arg \frac{\tilde{x}_2 y_1}{y_2} = 0\}$ and $\{\arg \frac{\tilde{x}_2 y_1}{y_2} = \pi\}$ the component $K_0' = \{\tilde{x}_2 y_1 + y_2 = 0\}$.

Similarly, we read off from (5.3) that the chain $f'^*\beta_2 \in C_4^{\mathrm{BM}}(\{\tilde{x}_2 = 0\}) \otimes \mathbb{Z}/2$ is (on a non-empty Zariski open subset) the boundary of

$$\gamma_2 := \mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) \cap \mathcal{C}\left(\frac{t_{23}}{(\tilde{x}_0 + \tilde{x}_1)t_{01}}\right) .$$

The boundary of $\gamma_2 \in C_5^{\mathrm{BM}}(\{\tilde{x}_2 = 0\}) \otimes \mathbb{Z}/2$ along the edges $\{\tilde{x}_2 = t_{01} = 0\}$ and $\{\tilde{x}_2 = t_{23} = 0\}$ is given by

$$\mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) .$$

On $\{\tilde{x}_2 = \tilde{x}_0 = 0\}$ and $\{\tilde{x}_2 = \tilde{x}_1 = 0\}$, the restriction of γ_2 is

$$\mathcal{C}\left(\frac{t_{23}}{t_{01}}\right) .$$

On $\{\tilde{x}_2 = \tilde{x}_0 + \tilde{x}_1 = 0\}$, we get

$$\mathcal{Q}\left(-1, \frac{\tilde{x}_0 y_0}{y_2}\right) ,$$

which is K_2'. Again, there is no boundary along $\{\tilde{x}_2 = y_0 = 0\}$ and $\{\tilde{x}_2 = y_2 = 0\}$.

Finally, in light of (5.4), the chain $f'^*\beta_{01} \in C_4^{\mathrm{BM}}(\{t_{01} = 0\}) \otimes \mathbb{Z}/2$ is the boundary of

$$\gamma_{01} := \mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) \cap \mathcal{C}\left(\frac{\tilde{x}_3}{\tilde{x}_2}\right) + \mathcal{C}\left(\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2\right) \cap \mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right) .$$

Its boundary along $\{t_{01} = \tilde{x}_0 = 0\}$ is

$$\mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right)$$

and its boundary along $\{t_{01} = \tilde{x}_2 = 0\}$ is

$$\mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) .$$

We see that the chains γ_0, γ_2, and γ_{01} have matching boundaries on the three considered edges $\{\tilde{x}_0 = t_{01} = 0\}$, $\{\tilde{x}_2 = t_{01} = 0\}$, and $\{x_0 = x_2 = 0\}$, and no further boundary components except K_i'. Moreover, their common restriction $\mathcal{C}(t_{23}/t_{01})$ onto the edge $\{\tilde{x}_0 = \tilde{x}_2 = 0\}$ does not contain the (non-real) points blown up by $Z \to Z'$.

Together with analogous computations for

$$\begin{aligned}
\gamma_1 &:= \mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right) \cap \mathcal{C}\left(\frac{(\tilde{x}_2 + \tilde{x}_3) t_{23}}{t_{01}}\right) , \\
\gamma_3 &:= \mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) \cap \mathcal{C}\left(\frac{t_{23}}{(\tilde{x}_0 + \tilde{x}_1) t_{01}}\right) , \\
\gamma_{23} &:= \ \mathcal{Q}\left(\frac{\tilde{x}_1}{\tilde{x}_0}, \frac{\tilde{x}_0 y_0}{y_2}\right) \cap \mathcal{C}\left(\frac{\tilde{x}_3}{\tilde{x}_2}\right) \ + \ \mathcal{C}\left(\frac{\tilde{x}_1}{\tilde{x}_0}\left(\frac{\tilde{x}_0 y_0}{y_2}\right)^2\right) \cap \mathcal{Q}\left(\frac{\tilde{x}_3}{\tilde{x}_2}, \frac{\tilde{x}_2 y_1}{y_2}\right)
\end{aligned}$$

we obtain the 5-chain

$$\gamma := \gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_{01} + \gamma_{23} \in C_5^{\mathrm{BM}}(W') \otimes \mathbb{Z}/2$$

with boundary $f'^*\beta + \kappa$. Its support is disjoint from the points blown up by $Z \to Z'$. This concludes the proof of Proposition 5.10. $\square$

Remark 5.12. At least on the four sides $\{\tilde{x}_i = 0\}$ of the cube, we can also describe topologically how the 5-chain γ arises:

For example, let us consider the divisor $\{\tilde{x}_0 = 0\}$. As we can see from the defining equation, the conic bundle $X' \to S'$ degenerates over $\{\tilde{x}_0 = 0\}$ into the union of two transversely intersecting lines on $\{\tilde{x}_2\tilde{x}_3 \neq 0\}$ and into a single non-reduced line on $\{\tilde{x}_2\tilde{x}_3 = 0\}$. Now β_0 is just the real square described by $\tilde{x}_1/\tilde{x}_0 \in [0, \infty]$ and $t_{23}/t_{01} \in [0, \infty]$ (where the expression t_{23}/t_{01} is well-defined only up to multiplication with $\tilde{x}_1/\tilde{x}_0$). As β_0 is simply connected, we can consistently pick one of the two lines in the fibre at each point in β_0. The idea is to "rotate" the segment $[0, \infty] \subset \mathbb{P}^1(\mathbb{C})$ (the one that describes the allowed values of $\tilde{x}_1/\tilde{x}_0$) around the origin until it comes back to itself. Due to the monodromy of the fibres, we arrive at the other line, so the boundary of the 5-chain swept out by this process is exactly the pullback $f'^*\beta_0$.

This phenomenon is similar to [AM72, § 2]. On the exceptional divisors $\{t_{01} = 0\}$ and $\{t_{23} = 0\}$, however, the conic bundle has smooth fibres in general and there does not seem to be a similar description with monodromy.

Bibliography

[Abh66] S. S. Abhyankar. *Resolution of singularities of embedded algebraic surfaces.* Number 24 in Pure and Applied Mathematics. Academic Press, 1966.

[AH61] M. F. Atiyah and F. Hirzebruch. Analytic cycles on complex manifolds. *Topology*, 1:25–45, 1961.

[AM72] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc.*, 25(3):75–95, 1972.

[Ara16] D. Arapura. Geometric Hodge structures with prescribed Hodge numbers. In M Kerr and G Pearlstein, editors, *Recent Advances in Hodge Theory*, number 427 in London Mathematical Society Lecture Note Series, pages 414–421. Cambridge University Press, 2016.

[AZ17] P. Achinger and M. Zdanowicz. Some elementary examples of non-liftable varieties. *Proc. Amer. Math. Soc.*, 145(11):4717–4729, 2017.

[BE10] R. Beheshti and D. Eisenbud. Fibers of generic projections. *Compos. Math.*, 146:435–456, 2010.

[BO20] O. Benoist and J. C. Ottem. Failure of the integral Hodge conjecture for threefolds of Kodaira dimension zero. *Comment. Math. Helv.*, 95:27–35, 2020.

[Buc49] A. Buchstab. On those numbers in an arithmetic progression all prime factors of which are small in order of magnitude. *Dokl. Akad. Nauk SSSR*, 67:5–8, 1949.

[BW20a] O. Benoist and O. Wittenberg. On the integral Hodge conjecture for real varieties, I. *Invent. Math.*, 222:1–77, 2020.

[BW20b] O. Benoist and O. Wittenberg. On the integral Hodge conjecture for real varieties, II. *J. Éc. polytech. Math.*, 7:373–429, 2020.

[CR11] A. Chatzistamatiou and K. Rülling. Higher direct images of the structure sheaf in positive characteristic. *Algebra Number Theory*, 5(6):693–775, 2011.

[CT95] J.-L. Colliot-Thélène. Birational invariants, purity and the gersten conjecture. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras*, number 58 in Proc. Sympos. Pure Math., pages 1–64. AMS, 1995.

[CTO89] J.-L. Colliot-Thélène and M. Ojanguren. Variétés unirationnelles non rationnelles: au-delà de l'exemple d'Artin et Mumford. *Invent. Math.*, 97:141–158, 1989.

[CTV12] J.-L. Colliot-Thélène and C. Voisin. Cohomologie non ramifiée et conjecture de Hodge entière. *Duke Math. J.*, 161(5):735–801, 2012.

[Cut09] S. D. Cutkosky. Resolution of singularities for 3-folds in positive characteristic. *Amer. J. Math.*, 131(1):59–127, 2009.

[DHS94] O. Debarre, K. Hulek, and J. Spandaw. Very ample linear systems on abelian varieties. *Math. Ann.*, 300:181–202, 1994.

[Dia20] H. Diaz. On the unramified cohomology of certain quotient varieties. *Math. Z.*, 296:261–273, 2020.

[Dic30] K. Dickman. On the frequency of numbers containing prime factors of a certain relative magnitude. *Arkiv för Matematik, Astronomi och Fysik*, 22A(10):1–14, 1930.

[Ful98] W. Fulton. *Intersection theory.* Springer, 1998.

[GH85] P. Griffiths and J. Harris. On the Noether–Lefschetz theorem and some remarks on codimension two cycles. *Math. Ann.*, 271:31–51, 1985.

[Gra04] C. Grabowski. *On the integral Hodge conjecture for 3-folds.* PhD thesis, Duke University, 2004.

[Gro85] M. Gros. Classes de Chern et classes de cycles en cohomologie de Hodge–Witt logarithmique. *Mém. Soc. Math. France (N. S.)*, 21:1–87, 1985.

[Hun89] B. Hunt. Complex manifold geography in dimension 2 and 3. *J. Differential Geom.*, 30(1):51–153, 1989.

[K$^+$91] J. Kollár et al. Trento examples. In E Ballico, F Catanese, and C Ciliberto, editors, *Classification of Irregular Varieties*, number 1515 in Lecture Notes in Mathematics, pages 134–139. Springer, 1991.

[KS13] D. Kotschick and S. Schreieder. The Hodge ring of Kähler manifolds. *Compos. Math.*, 149:637–657, 2013.

[Lef24] S. Lefschetz. *L'analysis situs et la géométrie algébrique.* Gauthier-Villars, 1924.

[Mat73] J. N. Mather. Generic projections. *Ann. of Math.*, 98(2):226–245, 1973.

[Mer74] F. Mertens. Ein Beitrag zur analytischen Zahlentheorie. *J. Reine Angew. Math.*, 78:46–62, 1874.

[OS20] J. C. Ottem and F. Suzuki. A pencil of Enriques surfaces with non-algebraic integral Hodge classes. *Math. Ann.*, 377:183–197, 2020.

[Pau22] M. Paulsen. On the degree of algebraic cycles on hypersurfaces. *J. Reine Angew. Math.*, 790:137–148, 2022.

[PS19] M. Paulsen and S. Schreieder. The construction problem for Hodge numbers modulo an integer. *Algebra Number Theory*, 13(10):2427–2434, 2019.

[Ray78] M. Raynaud. Contre-exemple au "vanishing theorem" en caractéristique $p > 0$. In *C. P. Ramanujam—a tribute*, number 8 in Tata Inst. Fund. Res. Studies in Math., pages 273–278. Springer, 1978.

[Sar82] V. Sarkisov. On conic bundle structures. *Math. USSR-Izv.*, 20(2):355–390, 1982.

[Sch15] S. Schreieder. On the construction problem for Hodge numbers. *Geom. Topol.*, 19:295–342, 2015.

[Sch19] S. Schreieder. Stably irrational hypersurfaces of small slopes. *J. Amer. Math. Soc.*, 32(4):1171–1199, 2019.

[Sch23] S. Schreieder. Refined unramified cohomology of schemes. *Compos. Math.*, 159(7):1466–1530, 2023.

[Ser58] J.-P. Serre. Sur la topologie des variétés algébriques en caractéristique p. In *Symposium internacional de topología algebraica*, pages 24–53. Universidad Nacional Autónoma de México and UNESCO, 1958.

[Sim04] C. Simpson. The construction problem in Kähler geometry. In S Donaldson, Y Eliashberg, and M Gromov, editors, *Different Faces of Geometry*, volume 3 of *International Mathematical Series*, pages 365–402. Springer, 2004.

[SV05] C. Soulé and C. Voisin. Torsion cohomology classes and algebraic cycles on complex projective manifolds. *Adv. Math.*, 198(1):107–127, 2005.

[Tot97] B. Totaro. Torsion algebraic cycles and complex cobordism. *J. Amer. Math. Soc.*, 10(2):467–493, 1997.

[Tot13] B. Totaro. On the integral Hodge and Tate conjectures over a number field. *Forum Math. Sigma*, 1(4):1–13, 2013.

[Tot21] B. Totaro. The integral Hodge conjecture for 3-folds of Kodaira dimension zero. *J. Inst. Math. Jussieu*, 20(5):1697–1717, 2021.

[vDdB21] R. van Dobben de Bruyn. The Hodge ring of varieties in positive characteristic. *Algebra Number Theory*, 15(3):729–745, 2021.

[vDdBP20] R. van Dobben de Bruyn and M. Paulsen. The construction problem for Hodge numbers modulo an integer in positive characteristic. *Forum Math. Sigma*, 8, 2020. no. e45.

[Voe00] V. Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, number 143 in Ann. of Math. Stud., pages 188–238. Princeton University Press, 2000.

[Voe03] V. Voevodsky. Motivic cohomology with $\mathbb{Z}/2$-coefficents. *Publ. Math. Inst. Hautes Études Sci.*, 98:59–104, 2003.

[Voi89] C. Voisin. Sur une conjecture de Griffiths et Harris. In E Ballico and C Ciliberto, editors, *Algebraic Curves and Projective Geometry*, number 1389 in Lecture Notes in Mathematics, pages 270–275. Springer, 1989.

[Voi03] C. Voisin. *Hodge Theory and Complex Algebraic Geometry I.* Number 76 in Cambridge studies in advanced mathematics. Cambridge University Press, 2003.

[Voi06] C. Voisin. On integral Hodge classes on uniruled or Calabi–Yau threefolds. In *Moduli Spaces and Arithmetic Geometry (Kyoto, 2004)*, number 45 in Advanced Studies in Pure Mathematics, pages 43–73, 2006.

[Wu90] X. Wu. On a conjecture of Griffiths–Harris generalizing the Noether–Lefschetz theorem. *Duke Math. J.*, 60(2):465–472, 1990.