Intersection Theory

Sheet 3

will be discussed on May 16

Exercise 1. Consider the cuspidal curve $X = \{y^2 = x^3\} \subset \mathbb{A}^2_k$ with cusp at $p = (0, 0)$.

(a) Show that $k(X)^*/\mathcal{O}^*_{X,p} \cong \mathbb{Z} \oplus k$.

(b) Show that ord_p corresponds to the projection $\mathbb{Z} \oplus k \to \mathbb{Z}$.

(c) Deduce that $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is surjective, with kernel isomorphic to k.

(d) Conclude that $\operatorname{Pic}(X) \cong k$.

Exercise 2. Let $X = \{x_0^2 = x_1x_2\} \subset \mathbb{P}^3$. Consider the lines $L_1 = \{x_0 = x_1 = 0\} \subset X$ and $L_2 = \{x_0 = x_2 = 0\} \subset X$ meeting at the singular point $[0 : 0 : 0 : 1] \in X$.

(a) Show that $2L_1$ is a Cartier divisor.

(b) Compute $2L_1 \cdot L_2 \in \operatorname{CH}_0(L_1 \cap L_2)$ and $2L_1 \cdot L_1 \in \operatorname{CH}_0(L_1)$.

Exercise 3. Let X be a variety. Let C be a Cartier divisor that represents a pseudo-divisor D on X. Let $\alpha \in \mathcal{Z}_k(X)$. Show that $C \cdot \alpha = D \cdot \alpha$ in $\operatorname{CH}_{k-1}(|D| \cap |\alpha|)$.

Exercise 4. Let S be a smooth surface, so we have $\mathcal{Z}_1(S) \cong \operatorname{CaDiv}(S)$. Let $C, D \subset S$ be curves (i. e. 1-dimensional subvarieties, but we may also regard them as effective Cartier divisors). Show that $C \cdot D \in \operatorname{CH}_0(C \cap D)$ is the cycle associated to the subscheme $C \cap D$ if $C \neq D$. Deduce that $C \cdot D = D \cdot C$.

Exercise 5. Prove the following version of Bezout's theorem: For curves $C_1, C_2 \subset \mathbb{P}^2$ of degrees d_1 and d_2, we have $\deg(C_1 \cdot C_2) = d_1d_2$.