Intersection Theory

Solution to sheet 3

will be discussed on May 16

Exercise 1. Consider the cuspidal curve $X = \{y^2 = x^3\} \subset \mathbb{A}^2_k$ with cusp at $p = (0,0)$.

(a) Show that $k(X)^*/\mathcal{O}^*_{X,p} \cong \mathbb{Z} \oplus k$.

(b) Show that ord_p corresponds to the projection $\mathbb{Z} \oplus k \to \mathbb{Z}$.

(c) Deduce that $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is surjective, with kernel isomorphic to k.

(d) Conclude that $\operatorname{Pic}(X) \cong k$.

Proof. Let us first observe that $X = \operatorname{Spec} k[x,y]/(y^2 - x^3) = \operatorname{Spec} k[t^2,t^3]$ and thus $\mathcal{O}_{X,p} = k[t^2,t^3]_{(t^2,t^3)}$ and $k(X) = k(t)$ (this was already mentioned in the lecture).

(a) We claim that

$$\begin{aligned} \psi\colon \mathbb{Z} \oplus k &\to k(X)^*/\mathcal{O}^*_{X,p} \\ (m,a) &\mapsto t^m(1+at) \end{aligned}$$

is an isomorphism of abelian groups.

Let us first prove that ψ is a group homomorphism: For this, we need to show that

$$\frac{\psi(m,a)\cdot\psi(n,b)}{\psi(m+n,a+b)} \in \mathcal{O}^*_{X,p}\,.$$

But

$$\frac{\psi(m,a)\cdot\psi(n,b)}{\psi(m+n,a+b)} = \frac{(1+at)(1+bt)}{(1+(a+b)t)} = \frac{(1+at)(1+bt)(1-(a+b)t)}{(1+(a+b)t)(1-(a+b)t)}\,.$$

Multiplying out the numerator and denominator, we see that both are polynomials in t with non-vanishing constant term and vanishing linear term. Therefore, they lie in $k[t^2,t^3] \setminus (t^2,t^3)$ and thus in $\mathcal{O}^*_{X,p}$.

Let us now prove that ψ is injective. Suppose that $t^m(1+at) \in \mathcal{O}_{X,p}^*$. This means that $t^m(1+at) = \frac{f}{g}$ for some polynomials $f, g \in k[t^2, t^3] \setminus (t^2, t^3)$, i. e. $f(0) \neq 0 \neq g(0)$ and $f'(0) = 0 = g'(0)$. Evaluation at 0 shows $m = 0$. Hence, we have $f = (1+at)g$. Therefore, $f' = ag + g'$ and evaluation at 0 shows $a = 0$.

Let us finally prove that ψ is surjective. Since ψ is a group homomorphism, it suffices to show that every polynomial $f \in k[t] \setminus (0)$ is in the image of ψ, up to $\mathcal{O}_{X,p}^*$. Since t^m is in the image of ψ, we may assume $f(0) \neq 0$. Since $k^* \subset \mathcal{O}_{X,p}^*$, we may even assume $f(0) = 1$. Let $a = f'(0)$ be the linear coefficient of f. Then

$$\frac{f}{1+at} = \frac{(1-at)f}{1-a^2t^2} \in \mathcal{O}_{X,p}^*$$

because the linear coefficient of $(1-at)f$ is $f'(0) - a \cdot f(0) = 0$, so numerator and denominator lie in $k[t^2, t^3] \setminus (t^2, t^3)$. This proves surjectivity.

(b) We need to show that $\mathrm{ord}_p(t) = 1$ and $\mathrm{ord}_p(1+at) = 0$.

One possible approach is to compute these orders explicitly using the definition of ord_p. Since $t = \frac{t^3}{t^2}$ with $t^2, t^3 \in \mathcal{O}_{X,p}$, we have

$$\mathrm{ord}_p(t) = \mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/t^3) - \mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/t^2) \ .$$

Now we use the fact that the length of these $\mathcal{O}_{X,p}$-modules agrees with their dimension as k-vector spaces (see Example A.1.1 in Fulton's book). One can check that $1, t^2, t^4$ forms a basis of $\mathcal{O}_{X,p}/t^3$, and $1, t^3$ forms a basis of $\mathcal{O}_{X,p}/t^2$. Therefore, $\mathrm{ord}_p(t) = 3 - 2 = 1$. To determine $\mathrm{ord}_p(1+at)$, let us write $1 + at = \frac{t^2+at^3}{t^2}$ where $t^2, t^2 + at^3 \in \mathcal{O}_{X,p}$, so we have

$$\mathrm{ord}_p(1+at) = \mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/(t^2+at^3)) - \mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/t^2) \ .$$

We have already seen that $\mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/t^2) = 2$. With a bit more work, one can also show $\mathrm{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/(t^2+at^3)) = 2$. Indeed, in $\mathcal{O}_{X,p}$ we have

$$(t^2+at^3)(t^2-at^3) = t^4 - a^2t^6 = t^4(1-a^2t^2) \ .$$

Since $1 - a^2t^2$ is a unit, this shows that $t^4 = 0$ in the quotient $\mathcal{O}_{X,p}/(t^2+at^3)$. Using this, one can check that $1, t^3$ forms a basis of $\mathcal{O}_{X,p}/(t^2+at^3)$. Consequently, we have $\mathrm{ord}_p(1+at) = 2 - 2 = 0$.

An alternative approach is to use the normalization

$$\begin{aligned} \nu \colon \mathbb{A}^1 &\to X \\ t &\mapsto (t^2, t^3) \ . \end{aligned}$$

Since the norm of the field extension $k(X) \subset k(\mathbb{A}^1)$ is the identity, the proof of $\nu_* \mathcal{Z}_0(\mathbb{A}^1)_{\mathrm{rat}} \subset \mathcal{Z}_0(X)_{\mathrm{rat}}$ (see Theorem 2.21 in the lecture notes) shows that

$$\mathrm{div}(t) = \nu_* \,\mathrm{div}(t) = \nu_* 0 = [k : k] \cdot \nu(0) = 1 \cdot p \ ,$$

so $\mathrm{ord}_p(t) = 1$. Similarly, we have $\mathrm{ord}_p(1+at) = 0$ because $\mathrm{ord}_0(1+at) = 0$ in the discrete valuation ring $\mathcal{O}_{\mathbb{A}^1,0}$.

(c) We have proven in the lecture (Lemma 3.4 in the lecture notes) that

$$\operatorname{CaDiv}(X) \cong \bigoplus_{q \in X} k(X)^*/\mathcal{O}^*_{X,q}$$

where the direct sum is taken over all closed points $q \in X$. Furthermore, the map $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is the direct sum of the maps $k(X)^*/\mathcal{O}^*_{X,q} \to \mathbb{Z}$ induced by ord_q. Since X is smooth at any closed point $q \neq p$, the local ring $\mathcal{O}_{X,q}$ is regular and thus a discrete valuation ring. Hence, ord_q induces an isomorphism $k(X)^*/\mathcal{O}^*_{X,q} \cong \mathbb{Z}$ for all $q \neq p$. Therefore, up to isomorphism, the map $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is the direct sum of the identity maps $\mathbb{Z} \to \mathbb{Z}$ for $q \neq p$ and of the projection $\mathbb{Z} \oplus k \to \mathbb{Z}$ from part (b). Thus $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ is surjective with kernel k.

(d) The map $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is obtained from $\operatorname{CaDiv}(X) \to \operatorname{Div}(X)$ by quotienting out the subgroup $\{\operatorname{div}(f) \mid f \in k(X)^*\}$ on both sides. Hence, $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is surjective as well. We claim that its kernel is still k. For this, we need to show for $f \in k(X)^* = k(t)^*$ that if $\operatorname{div}(f) = 0$ as a Weil divisor (i. e. $\operatorname{ord}_q(f) = 0$ for all closed points $q \in X$), then also $\operatorname{div}(f) = 0$ as a Cartier divisor (i. e. $f \in \mathcal{O}^*_{X,q}$ for all closed points $q \in X$). Since $\mathcal{O}_{X,q}$ is a discrete valuation ring for all $q \neq p$, we have $f \in \mathcal{O}^*_{X,q}$ for all $q \neq p$. Let us write $f = \alpha g_1^{m_1} \cdots g_r^{m_r}$ with $\alpha \in k^*$, irreducible monic polynomials $g_1, \dots, g_r \in k[t]$, and exponents $m_1, \dots, m_r \in \mathbb{Z}$. By considering the closed points in $\mathbb{A}^1 \setminus \{0\} \cong X \setminus \{p\}$ corresponding to $g_i \neq t$, we see that $f = \alpha t^m$ for some $m \in \mathbb{Z}$. Since $\operatorname{ord}_p(t) = 1$, we conclude that $m = 0$ and thus $f \in \mathcal{O}^*_{X,p}$.

(Remark: For arbitrary varieties X, it is not true that $\operatorname{div}(f) = 0$ in $\operatorname{Div}(X)$ automatically implies $\operatorname{div}(f) = 0$ in $\operatorname{CaDiv}(X)$. For example, it is wrong for $f = 1 - t$ if we remove the smooth point $(1, 1)$ from our cuspidal curve X.)

Finally, we have $\operatorname{Cl}(X) = 0$: This is because $p = \operatorname{div}(t)$ and $q = \operatorname{div}(g)$ for a closed point $q \neq p$ where $g \in k[t]$ corresponds to $q \in X \setminus \{p\} \cong \mathbb{A}^1 \setminus \{0\}$. Therefore, $\operatorname{Pic}(X) \cong k$. □

Exercise 2. Let $X = \{x_0^2 = x_1 x_2\} \subset \mathbb{P}^3$. Consider the lines $L_1 = \{x_0 = x_1 = 0\} \subset X$ and $L_2 = \{x_0 = x_2 = 0\} \subset X$ meeting at the singular point $[0 : 0 : 0 : 1] \in X$.

(a) Show that $2L_1$ is a Cartier divisor.

(b) Compute $2L_1 \cdot L_2 \in \mathrm{CH}_0(L_1 \cap L_2)$ and $2L_1 \cdot L_1 \in \mathrm{CH}_0(L_1)$.

Proof. (a) The subscheme $D = \{x_1 = 0\} \subset X$ is defined by a single homogeneous polynomial and can thus be described locally by a single equation. Conretely, D is defined on the affine open subset $U_i = \{x_i \neq 0\} \subset X$ by $\frac{x_1}{x_i} \in \mathcal{O}_X(U_i)$. Hence, D is an effective Cartier divisor. Since L_1 is the reduction of D, the Weil divisor associated to D is given by mL_1 where $m = \mathrm{length}_{\mathcal{O}_{D,L_1}}(\mathcal{O}_{D,L_1})$. We have

$$D \cap U_3 = \operatorname{Spec} k[t_0, t_1, t_2]/(t_0^2 - t_1 t_2, t_1) = \operatorname{Spec} k[t_0, t_2]/(t_0^2)$$

where $t_j = \frac{x_j}{x_3}$. Since the generic point of L_1 is contained in U_3, it follows that

$$\mathcal{O}_{D,L_1} = \left(k[t_0, t_2]/(t_0^2)\right)_{(t_0)} ,$$

which has length 2 (a maximal chain of ideals is $(0) = (t_0^2) \subsetneq (t_0) \subsetneq (1)$). Therefore, $D = 2L_1$. Hence, $2L_1$ is a Cartier divisor.

(b) By definition, $2L_1 \cdot L_2$ is the Weil divisor class of the pullback (as a pseudo-divisor) of $2L_1$ to L_2. In this case, $2L_1$ pulls back to L_2 already as an effective Cartier divisor because the regular functions $\frac{x_1}{x_i} \in \mathcal{O}_X(U_i)$ defining $2L_1$ do not completely vanish on L_2. Therefore, $2L_1 \cdot L_2$ is the cycle class of the 0-dimensional subscheme $\{x_0 = x_2 = 0\} \cap \{x_1 = 0\} = \{x_0 = x_1 = x_2 = 0\}$. In other words, $2L_1 \cdot L_2 = [p] \in \mathrm{CH}_0(L_1 \cap L_2)$ where $p = [0 : 0 : 0 : 1]$.

On the other hand, $2L_1$ does not pull back to L_1 as a Cartier divisor. Instead the pullback will be a pseudo-divisor with support $|2L_1| \cap L_1 = L_1$. Hence, its cycle class in $\mathrm{CH}_0(L_1)$ is given by the Weil divisor class of any Cartier divisor C on L_1 such that $\mathcal{O}_{L_1}(C) \cong \mathcal{O}_X(2L_1)|_{L_1}$. Note that $\operatorname{div}\left(\frac{x_1}{x_2}\right) = 2L_1 - 2L_2$ in $\mathrm{CaDiv}(X)$ (here we use that $2L_2$ agrees with the effective Cartier divisor $\{x_2 = 0\}$ by exactly the same argument as for $2L_1$). Therefore, $\mathcal{O}_X(2L_1) \cong \mathcal{O}_X(2L_2)$. Since $2L_2$ restricts to a Cartier divisor on L_1 (this is the same situation as previously, with L_1 and L_2 interchanged), we thus have $\mathcal{O}_X(2L_1)|_{L_1} = \mathcal{O}_{L_1}(2L_2)$. In other words, we can take $C = 2L_2$ as a representing Cartier divisor. Hence, $2L_1 \cdot L_1$ is the cycle class of the 0-dimensional subscheme $\{x_0 = x_1 = 0\} \cap \{x_2 = 0\} = \{x_0 = x_1 = x_2 = 0\}$, so $2L_1 \cdot L_1 = [p] \in \mathrm{CH}_0(L_1)$. □

Exercise 3. Let X be a variety. Let C be a Cartier divisor that represents a pseudo-divisor D on X. Let $\alpha \in \mathcal{Z}_k(X)$. Show that $C \cdot \alpha = D \cdot \alpha$ in $\mathrm{CH}_{k-1}(|D| \cap |\alpha|)$.

Proof. By linearity, it suffices to show $C \cdot Z = D \cdot Z$ for a k-dimensional subvariety $Z \subset X$. By definition, $C \cdot Z = [i^*C]$ and $D \cdot Z = [i^*D]$ where $i\colon Z \hookrightarrow X$ denotes the inclusion. However, $[i^*C]$ and $[i^*D]$ are the Weil divisor classes associated to a Cartier divisor representing the pseudo-divisor i^*C and i^*D, respectively. Since C represents D, we have

$$|i^*C| = |C| \cap Z \subset |D| \cap Z = |i^*D|$$

and

$$\mathcal{O}_Z(i^*C) = i^*\mathcal{O}_X(C) \cong i^*\mathcal{O}_X(D) = \mathcal{O}_Z(i^*D) \ .$$

Additionally, under this isomorphism we have

$$s_{i^*C}|_{Z \setminus |i^*D|} = i^*(s_C|_{X \setminus |D|}) = i^*(s_D) = s_{i^*D} \ .$$

Therefore, any Cartier divisor representing i^*C also represents i^*D, and thus $[i^*C] = [i^*D]$. □

Exercise 4. Let S be a smooth surface, so we have $\mathcal{Z}_1(S) \cong \mathrm{CaDiv}(S)$. Let $C, D \subset S$ be curves (i. e. 1-dimensional subvarieties, but we may also regard them as effective Cartier divisors). Show that $C \cdot D \in \mathrm{CH}_0(C \cap D)$ is the cycle associated to the subscheme $C \cap D$ if $C \neq D$. Deduce that $C \cdot D = D \cdot C$.

Proof. First note that $C \cap D$ is 0-dimensional because $C \neq D$ are irreducible. Let $p \in C \cap D$ and consider the 2-dimensional local ring $A = \mathcal{O}_{S,p}$. Since C and D are effective Cartier divisors, they are described locally around p by equations $f, g \in A$. We have $\mathcal{O}_{C,p} = A/f$ and $\mathcal{O}_{D,p} = A/g$. Since $C \neq D$ are subvarieties of codimension 1, f and g generate prime ideals $(f) \neq (g)$ in A of height 1. In particular, f is non-zero in A/g, so C pulls back to a Cartier divisor on D (supported on $C \cap D$). By definition, $C \cdot D$ is the Weil divisor associated to this pullback. Its coefficient at the prime divisor $\{p\} \subset D$ is by definition the order of vanishing of $f \in \mathcal{O}_{D,p} = A/g$, i. e. the length of the (A/g)-module $A/(f, g)$.

On the other hand, the coefficient of p in the cycle associated to the subscheme $C \cap D$ is the length of $\mathcal{O}_{C \cap D,p} = A/(f, g)$ over itself. Hence, it remains to show

$$\mathrm{length}_{A/g}(A/(f, g)) = \mathrm{length}_{A/(f,g)}(A/(f, g)) \ .$$

This is clear because every (A/g)-submodule of $A/(f, g)$ is annihilated by f and thus an ideal of $A/(f, g)$.

We proved $C \cdot D = [c(C \cap D)]$ for $C \neq D$. It follows that

$$C \cdot D = [c(C \cap D)] = [c(D \cap C)] = D \cdot C$$

if $C \neq D$. And if $C = D$, then $C \cdot D = D \cdot C$ is trivially true. □

Exercise 5. Prove the following version of Bezout's theorem: For curves $C_1, C_2 \subset \mathbb{P}^2$ of degrees d_1 and d_2, we have $\deg(C_1 \cdot C_2) = d_1 d_2$.

Proof. Let $L_1, L_2 \subset \mathbb{P}^2$ be distinct lines. Since $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}$, the Cartier divisors C_1 and $d_1 L_1$ both represent the pseudo-divisor given by the line bundle $\mathcal{O}_{\mathbb{P}^2}(C_1) = \mathcal{O}_{\mathbb{P}^2}(d_1 L_1)$. Hence, Exercise 3 implies $C_1 \cdot C_2 = d_1 L_1 \cdot C_2$ in $\mathrm{CH}_0(C_2)$. By linearity (Lemma 3.16 in the lecture notes), we have $d_1 L_1 \cdot C_2 = d_1(L_1 \cdot C_2)$. By the commutativity proven in Exercise 4, we have $L_1 \cdot C_2 = C_2 \cdot L_1$. Repeating the previous arguments for C_2 and L_2, we get $C_2 \cdot L_1 = d_2(L_1 \cdot L_2)$. Finally, since $L_1 \neq L_2$, by Exercise 4 we have $L_1 \cdot L_2 = [c(L_1 \cap L_2)] = [p]$ for some k-rational point $p \in \mathbb{P}^2$. Putting everything together, we obtain

$$\deg(C_1 \cdot C_2) = \deg(d_1(C_2 \cdot L_1)) = \deg(d_1 d_2(L_1 \cdot L_2)) = d_1 d_2 \deg(p) = d_1 d_2 \,. \qquad \square$$