Intersection Theory

Sheet 2

will be discussed on May 2

Exercise 1. Let $f\colon X \to Y$ be a proper morphism between proper algebraic schemes. Show that $\deg(f_*\alpha) = \deg(\alpha)$ for all $\alpha \in \mathrm{CH}_0(X)$.

Exercise 2. Let $X_1, X_2 \subset X$ be closed subsets of an algebraic scheme X, and consider the inclusions $i_t\colon X_1 \cap X_2 \to X_t$ and $j_t\colon X_t \to X_1 \cup X_2$ for $t \in \{1, 2\}$. Prove that the sequence

$$\mathrm{CH}_k(X_1 \cap X_2) \overset{i_{1,*}+i_{2,*}}{\to} \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \overset{j_{1,*}-j_{2,*}}{\to} \mathrm{CH}_k(X_1 \cup X_2) \to 0$$

is exact.

Exercise 3. In this exercise, you will prove $\mathrm{CH}_k(\mathbb{P}^n) \cong \mathbb{Z}$ (with generator $\mathbb{P}^k \subset \mathbb{P}^n$) for all $k \in \{0, \ldots, n\}$ via induction on n.

(a) Show the statement for $k = n$ and $k = n - 1$.

(b) From now on, let $k \leq n - 2$. Use the localization exact sequence, the induction hypothesis, and the fact from the lecture that $\mathrm{CH}_k(\mathbb{A}^n) = 0$ in order to prove that there exists a surjection $\mathbb{Z} \to \mathrm{CH}_k(\mathbb{P}^n)$ sending $1 \in \mathbb{Z}$ to the class of a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$.

(c) Show that for every closed subscheme $X \subset \mathbb{P}^n$ of pure dimension $k + 1$ containing a linear subspace $\mathbb{P}^k \subset X$, there exists a surjection $\mathrm{CH}_k(X) \to \mathbb{Z}$ sending $[\mathbb{P}^k]$ to $1 \in \mathbb{Z}$. (Hint: Use the pushforward to $\mathrm{CH}_k(\mathbb{P}^{k+1})$ via a suitable linear projection.)

(d) Conclude that $\mathrm{CH}_k(\mathbb{P}^n) \cong \mathbb{Z}$ (generated by a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$).

Exercise 4. Let $f\colon X \to Y$ be a flat morphism which is finite of degree d (in particular, f is proper). Show that $f_* f^* \alpha = d \cdot \alpha$ for all $\alpha \in \mathrm{CH}_k(Y)$.

Exercise 5. Show that $\mathcal{Z}_k(X)_{\mathrm{rat}}$ is the subgroup generated by cycles $Z_0 - Z_\infty \in \mathcal{Z}_k(X)$ for a $(k + 1)$-dimensional subvariety $Z \subset X \times \mathbb{P}^1$ that dominates $\mathbb{P}^1$. Here, $Z_t \subset X$ denotes the fibre at $t \in \mathbb{P}^1$. (As a consequence of this exercise, rational equivalence can be defined in an alternative way.)