Intersection Theory*

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*If you find any mistakes in these lecture notes, please let me know.

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Organisational stuff

Lecture The class is on Monday and Tuesday from 10am to 12pm in room g117. Since there is an overlap on Monday with the lecture on Algebraic Groups, I will see if it is possible to change the date. These lecture notes will be continuously updated. I periodically upload the most recent version to Stud.IP and to the website at `https://math.mp42.de/chow/`.

Exercises Roughly every second week, one of the two lectures will be replaced by an exercise session where we discuss the problems on the exercise sheet. The exercise sheet appears approximately one week before the exercise session and can be downloaded from Stud.IP or from the website at `https://math.mp42.de/chow/`. It is possible to hand in solutions, either personally to me (in the lecture or in my office g111) or by sending a PDF file to `paulsen@math.uni-hannover.de`.

Literature The main reference book on intersection theory is [Ful98]. Another good resource (but which develops intersection theory a bit differently than in this lecture) is [Sta22, Chapters 42 and 43]. A brief overview on intersection theory can be found in [Har77, Appendix A]. Some parts of the lecture will also be based on [Voi03, Part III]. A very nice book for intuition and for applications to enumerative geometry is [EH16].

Prerequisites You need to know basic algebraic geometry at least on the level of an introductory course "Algebraic Geometry I". The first chapter of [Har77] is a good reference here. If you don't know schemes and coherent sheaves yet, I highly recommend to take "Algebraic Geometry II" in parallel to this course, since in many places it won't be possible to avoid these notions. It is helpful (but not necessary) if you also have some knowledge in topology, such as singular cohomology. Please let me know if you don't know something that I use in the lecture. I'm happy to explain these things.

Feedback If you have any questions or suggestions, please don't hesitate to contact me, e. g. in the lecture, in my office g111, or via `paulsen@math.uni-hannover.de`.

Exam For those who want credits, I will organize an oral exam at the end of the semester. I will provide more details later.

1 Motivation and overview

This chapter tries to give an informal introduction to intersection theory. More precise definitions of the notions appearing here will be given later.

In the following, a *variety* is an integral separated scheme of finite type over a field. But if you don't like/know schemes, you could just think of quasi-projective varieties in the classical sense. Also, *subvarieties* are always closed.

Let us recall Bézout's theorem about curves in the projective plane:

Theorem 1.1 (Bézout)**.** *Let $C_1, C_2 \subset \mathbb{P}^2_{\mathbb{C}}$ be curves of degree d_1 and d_2, respectively. If C_1 and C_2 intersect transversely, then*

$$|C_1 \cap C_2| = d_1 \cdot d_2 \ .$$

Here, saying that C_1 and C_2 intersect *transversely* at some point p means that C_1 and C_2 are smooth at p and that the tangent directions of C_1 and C_2 at p are different, so T_pC_1 and T_pC_2 together span $T_p\mathbb{P}^2_{\mathbb{C}}$.

The important point in Bézout's theorem is that $|C_1 \cap C_2|$ does not depend on the specific curves, but only on their degrees[1]. If we "slightly move" C_1 or C_2, their intersection $C_1 \cap C_2$ "slightly moves" as well. In particular, the number of intersection points does not change.

Intersection theory tries to generalize this fact (i. e. that $C_1 \cap C_2$ is well-defined modulo "slight movements" of C_1 and C_2) from $\mathbb{P}^2_{\mathbb{C}}$ to other varieties X, and from curves to arbitrary subvarieties (or more generally, formal $\mathbb{Z}$-linear combinations of subvarieties).

One way to describe "slight movements" is the notion of *rational equivalence*: Roughly speaking, two subvarieties $Z, Z' \subset X$ of the same dimension are *rationally equivalent* if there exists a family of subvarieties $Z_t \subset X$ parametrized by $t \in \mathbb{P}^1$ containing both Z and Z'. If you know topology, this might look a bit like an algebraic version of a homotopy to you. A more precise definition of rational equivalence will be given later.

[1] If we know that, it already implies that this number must be $d_1 \cdot d_2$, since we can then just take C_i to be d_i general lines for $i \in \{1, 2\}$.

Example 1.2. For homogeneous polynomials $f, g \in \mathbb{C}[x_0, \ldots, x_n]$ of the same degree d, the hypersurfaces $Z = \{f = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$ and $Z' = \{g = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$ are rationally equivalent via the family

$$Z_t = \{f + tg = 0\} \subset \mathbb{P}^n_{\mathbb{C}}\ , \quad t \in \mathbb{P}^1_{\mathbb{C}}\ ,$$

since $Z_0 = Z$ and $Z_\infty = Z'$.

Remark 1.3. One could also look at coarser equivalence relations. For example, if we replace the base $\mathbb{P}^1$ of the family by an arbitrary curve C, we get *algebraic equivalence.* In some sense, rational equivalence is the finest equivalence relation that gives a somewhat managable quotient and allows intersection theory to work.

Let us denote by $[Z]$ the equivalence class of a subvariety $Z \subset X$ with respect to rational equivalence. Intersection theory now tries to associate to any equivalence classes $[Z_1]$ and $[Z_2]$ a new equivalence class $[Z_1] \cdot [Z_2]$ in a consistent way such that we have $[Z_1]\cdot[Z_2] = [Z_1 \cap Z_2]$ whenever Z_1 and Z_2 intersect *generically transversally* (this basically means that the condition of transversality is only required for general points in the irreducible components of the intersection).

For what varieties X might such a theory fail?

1. If we replace $\mathbb{P}^2_{\mathbb{C}}$ by $\mathbb{A}^2_{\mathbb{C}}$, two distinct lines in $\mathbb{A}^2_{\mathbb{C}}$ can either intersect in 1 point or, if they are parallel, in 0 points. But in both cases they intersect transversely. This might suggest that no reasonable intersection theory for $\mathbb{A}^2_{\mathbb{C}}$ exists. The problem here seems that $\mathbb{A}^2_{\mathbb{C}}$, in contrast to $\mathbb{P}^2_{\mathbb{C}}$, is not proper (i. e. non-compact in the Euclidean topology).

   However, it turns out that an intersection theory, in the sense as described above, is actually possible for $\mathbb{A}^2_{\mathbb{C}}$: Every subvariety $Z \subset \mathbb{A}^2_{\mathbb{C}}$ of dimension 0 or 1 is just rationally equivalent to the empty set. Hence, the above phenomenon does not produce a contradiction since the intersection of two non-parallel lines (a single point) is indeed rationally equivalent to the intersection of two parallel lines (the empty set).

   The problem with non-proper varieties is rather that we cannot "count points" modulo rational equivalence, meaning that we cannot assign an integer (called *degree*) to a formal $\mathbb{Z}$-linear combination of zero-dimensional subvarieties in such a way that it respects rational equivalence and sends rational points to 1. Hence, we cannot define intersection *numbers* but might still have a perfectly reasonable intersection theory.

2. If we replace $\mathbb{P}^2_{\mathbb{C}}$ by $\mathbb{P}^2_{\mathbb{R}}$, two conics in $\mathbb{P}^2_{\mathbb{R}}$ can intersect transversely in 0, 2, or 4 points (if we only consider $\mathbb{R}$-points). Hence, Bézout's theorem does not seem to be true anymore (at least not in a naive way). Here the problem is that $\mathbb{R}$, in contrast to $\mathbb{C}$, is not algebraically closed.

As we will see later, one can come around this problem by looking at the scheme-theoretic intersection of these conics. In addition to the 0, 2, or 4 real intersection points, there appear 2, 1, or 0 more closed points in the intersection. But their residue field is a finite extension of the ground field (it is $\mathbb{C}$ instead of $\mathbb{R}$). If we take the degree $[\mathbb{C} : \mathbb{R}] = 2$ into account when computing the degree of the intersection, we get back the expected and correct answer 4.

3. If we replace $\mathbb{P}^2_{\mathbb{C}}$ by the projective surface

$$X = \{x_0^2 = x_1 x_2\} \subset \mathbb{P}^3_{\mathbb{C}}$$

(a cone over a conic), singular at $[0 : 0 : 0 : 1]$, the lines

$$L_1 = \{x_0 = x_1 = 0\} \subset X$$

and

$$L_2 = \{x_0 = x_2 = 0\} \subset X$$

intersect the conic

$$H = \{x_3 = 0\} \cap X$$

transversally in $p_1 = [0 : 0 : 1 : 0]$ and $p_2 = [0 : 1 : 0 : 0]$, respectively. However, H is rationally equivalent to any other hyperplane section, such as

- $\{x_0 = 0\} \cap X = L_1 + L_2$,
- $\{x_1 = 0\} \cap X = 2 \cdot L_1$, or
- $\{x_2 = 0\} \cap X = 2 \cdot L_2$

(we are a bit sloppy here – the correct formalism for this kind of equations will be developed later). Hence, we would get

$$[p_2] = [H] \cdot [L_2] = 2 \cdot [L_1] \cdot [L_2] \,.$$

This is impossible because, as indicated in the first item, for proper varieties there is a well-defined degree map to $\mathbb{Z}$. But then $[L_1] \cdot [L_2]$ would need to have degree $\frac{1}{2}$.

The problem here is that X, in contrast to $\mathbb{P}^2_{\mathbb{C}}$, is not smooth (even though X is normal). This is really a problem. There is no easy way to fix it, except defining intersection theory only for smooth varieties.

Applications One interesting application of intersection theory lies in the area of enumerative geometry, which allows to answer questions such as:

- How many lines are on a given smooth cubic surface? (Answer: 27)
- How many lines are on a given general quintic hypersurface in $\mathbb{P}^4$? (Answer: 2875)
- How many smooth conics are tangent to five given general conics? (Answer: 3264)

- How many planes in $\mathbb{P}^3$ are tangent to a given smooth cubic surface and contain a given general line? (Answer: 12)

This is done by choosing our variety X to be the parameter space of the objects we want to count (in many cases, this is some Grassmannian) and to regard the additional constraints (such as being tangent to given objects) as subvarieties in X that we want to intersect.

2 Chow groups

As described in the introduction, we want to work with rational equivalence classes of $\mathbb{Z}$-linear combinations of subvarieties of X. For subvarieties of a given dimension k, these equivalence classes are the elements of the k-th *Chow group* of X. One can view Chow groups as a generalization of the (Weil) divisor class group of X (which is the case $k = \dim(X) - 1$).

In this chapter, we finally define rational equivalence precisely. Then we define Chow groups and develop their functorial properties, such as proper pushforward and flat pullback. Chow groups are the main protagonist of this course, and are (even without the intersection product) a very interesting subject for themselves. They are somehow an algebraic version of homology groups, but are much harder to compute than homology. There are still many unsolved questions about Chow groups.

2.1 Recap: Divisor class group

In this section, we briefly recall the class group of Weil divisors on a normal variety.

As in the introduction, a *variety* is an integral separated scheme of finite type over a field (usually denoted by k), and *subvarieties* are always closed.

Definition 2.1. Let X be a variety. A *prime divisor* on X is a subvariety of codimension 1. A *divisor* is a formal $\mathbb{Z}$-linear combination

$$m_1 D_1 + \cdots + m_r D_r$$

of prime divisors $D_1, \ldots, D_r \subset X$ with coefficients $m_1, \ldots, m_r \in \mathbb{Z}$. We denote by $\mathrm{Div}(X)$ the abelian group of divisors on X. In other words, $\mathrm{Div}(X)$ is the free abelian group over the prime divisors on X.

Divisors occur naturally as the zeros and poles of rational functions on X. Concretely, every non-zero rational function $f \in k(X)^*$ gives rise to a divisor $\mathrm{div}(f) \in \mathrm{Div}(X)$, called a *principal divisor*. To define $\mathrm{div}(f)$, we first need to define the order of vanishing of f at some prime divisor $D \subset X$. Note that $\mathcal{O}_{X,D}$ is a local ring of dimension 1 with fraction field $k(X)$. If we assume that X is normal, then $\mathcal{O}_{X,D}$ is regular. From commutative algebra we know that every regular local ring of dimension 1 is a discrete valuation ring. We denote the corresponding discrete valuation by

$$\mathrm{ord}_D\colon k(X)^* \to \mathbb{Z}\,.$$

Definition 2.2. Let X be a normal variety. For $f \in k(X)^*$, let

$$\mathrm{div}(f) = \sum_{D\subset X} \mathrm{ord}_D(f)D \in \mathrm{Div}\,X$$

where the sum is taken over all prime divisors $D \subset X$.

To check that $\mathrm{div}(f)$ is well-defined, we need to verify that for a given $f \in k(X)^*$ only finitely many prime divisors $D \subset X$ with $\mathrm{ord}_D(f) \neq 0$ exist: Since f is a rational function on X, there exists a non-empty open subset $U \subset X$ where f is defined, meaning that $f \in \mathcal{O}_X(U)$. Since $f \not\equiv 0$, we have $f \in \mathcal{O}_X^*(V)$ for the non-empty open subset $V = U \setminus \{f = 0\}$. But this implies that f is a unit inside $\mathcal{O}_{X,D}$ for all prime divisors $D \subset X$ such that $V \cap D \neq \emptyset$. Therefore, we can have $\mathrm{ord}_D(f) \neq 0$ only if $D \subset X \setminus V$. Since X is irreducible and $V \neq \emptyset$, we have $\dim(X \setminus V) < \dim(X)$. Thus a prime divisor $D \subset X$ with $\mathrm{ord}_D(f) \neq 0$ needs to be an irreducible component of $X \setminus V$, of which only finitely many exist.

By the properties of discrete valuations,

$$\mathrm{div}\colon k(X)^* \to \mathrm{Div}\,X$$

is a morphism of abelian groups. The divisor class group of X is defined as the cokernel of this map:

Definition 2.3. The quotient

$$\mathrm{Cl}(X) = \frac{\mathrm{Div}(X)}{\{\mathrm{div}(f) \mid f \in k(X)^*\}}$$

is the divisor class group of X.

An important result about $\mathrm{Cl}(X)$ is that if X is sufficiently nice, the divisor class group $\mathrm{Cl}(X)$ agrees with the Picard group $\mathrm{Pic}(X)$, i. e. the group of isomorphism classes of line bundles on X, with composition given by the tensor product $\otimes$:

Theorem 2.4. *If X is locally factorial, then* $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$.

In particular, we have $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ if X is smooth.

2.2 Order of vanishing

In the previous section, the definition of $\operatorname{div}(f)$ assumed X to be normal, in order to have a discrete valuation ring $\mathcal{O}_{X,D}$ for each prime divisor $D \subset X$. We want to generalize this definition to arbitrary varieties X. For this we need to define the order of vanishing $\operatorname{ord}_D(f)$ of a non-zero rational function $f \in k(X)^*$ at a prime divisor $D \subset X$ even if the local ring $\mathcal{O}_{X,D}$ is not regular.

Recall that the *length* of a module M over a ring A is defined, in analogy to the dimension of a vector space over a field, as the maximum integer r such that there exists a chain

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M$$

of A-modules. If no maximal r exists, we set $\operatorname{length}_A(M) = \infty$.

Lemma 2.5. *We have* $\operatorname{length}_A(A/f) < \infty$ *for all* $f \in A \setminus \{0\}$ *if* $A = \mathcal{O}_{X,D}$ *is the local ring at a prime divisor* $D \subset X$.

Proof. If $f \in A^*$ is a unit, we clearly have $\operatorname{length}_A(A/f) = \operatorname{length}_A(0) = 0 < \infty$. Now let $f \notin A^*$. Since any A-submodule of A/f is annihilated by f and is thus also an (A/f)-module, it suffices to show $\operatorname{length}_{A/f}(A/f) < \infty$. This is equivalent to A/f being an artinian ring. As we know that A/f is noetherian, it remains to show $\dim(A/f) = 0$. However, the prime ideals of A/f correspond precisely to the prime ideals of A containing f. The only prime ideals of A are $\{0\}$ and the maximal ideal, because A is an integral domain, local, and of dimension 1. Since $f \neq 0$, it follows that the maximal ideal of A/f is the only prime ideal. □

Definition 2.6. Let $D \subset X$ be a prime divisor and consider the 1-dimensional local ring $A = \mathcal{O}_{X,D}$ with fraction field $k(X)$. For a non-zero rational function $f \in k(X)^*$ we define

$$\operatorname{ord}_D(f) = \operatorname{length}_A(A/g) - \operatorname{length}_A(A/h)$$

where $f = \frac{g}{h}$ with $g, h \in A \setminus \{0\}$.

Lemma 2.7. *The map*

$$\operatorname{ord}_D \colon k(X)^* \to \mathbb{Z}$$

is well-defined and a morphism of abelian groups.

Proof. Both statements are an immediate consequence of the formula

$$\operatorname{length}_A(A/ab) = \operatorname{length}_A(A/a) + \operatorname{length}_A(A/b)$$

for $a, b \in A$. This formula follows from the short exact sequence of A-modules

$$0 \to A/a \xrightarrow{\cdot b} A/ab \to A/b \to 0$$

and from the additivity of length in short exact sequences. □

Lemma 2.8. *If $A = \mathcal{O}_{X,D}$ is a discrete valuation ring, then* ord_D *agrees with the previous definition (i. e. with the discrete valuation).*

2.3 Definition of Chow groups

An *algebraic scheme* is a separated scheme of finite type over a field.

Definition 2.9. Let X be an algebraic scheme. A *k-cycle* on X is a formal $\mathbb{Z}$-linear combination

$$m_1 Z_1 + \cdots + m_r Z_r$$

of k-dimensional subvarieties $Z_1, \ldots, Z_r \subset X$ with coefficients $m_1, \ldots, m_r \in \mathbb{Z}$. We denote by $\mathcal{Z}_k(X)$ the abelian group of k-cycles on X. In other words, $\mathcal{Z}_k(X)$ is the free abelian group over the k-dimensional subvarieties of X.

Definition 2.10. For a closed subscheme $Y \subset X$ of dimension $\leq k$ its associated k-cycle is

$$c(Y) = \sum_{Z \subset Y} \operatorname{length}_{\mathcal{O}_{Y,Z}}(\mathcal{O}_{Y,Z}) Z \in \mathcal{Z}_k(X)$$

where the sum is taken over the irreducible k-dimensional components of Y.

Definition 2.11. For a $(k+1)$-dimensional subvariety $W \subset X$ and a non-zero rational function $f \in k(W)^*$, let

$$\operatorname{div}(f) = \sum_{Z \subset W} \operatorname{ord}_Z(f) Z \in \mathcal{Z}_k(X)$$

where the sum is taken over all k-dimensional subvarieties $Z \subset W$.

As in section 2.1, we need to ensure that $\operatorname{ord}_Z(f) = 0$ for all but finitely many prime divisors $Z \subset W$. By the same argument as before, $f \in \mathcal{O}_{W,Z}$ is a unit for all but finitely many prime divisors $Z \subset W$. However, this implies $\operatorname{ord}_Z(f) = \operatorname{length}_{\mathcal{O}_{W,Z}}(0) = 0$.

Definition 2.12. Let $\mathcal{Z}_k(X)_{\mathrm{rat}} \subset \mathcal{Z}_k(X)$ denote the abelian subgroup generated by $\operatorname{div}(f)$ for all $W \subset X$ and all $f \in k(W)^*$ as above. We define

$$\mathrm{CH}_k(X) = \frac{\mathcal{Z}_k(X)}{\mathcal{Z}_k(X)_{\mathrm{rat}}}$$

to be the *k-th Chow group* of X. If X is irreducible of dimension n, we set

$$\mathrm{CH}^k(X) = \mathrm{CH}_{n-k}(X) \ .$$

For a cycle $\alpha \in \mathcal{Z}_k(X)$, we denote the corresponding equivalence class by $[\alpha] \in \mathrm{CH}_k(X)$.

Definition 2.13. Two k-cycles $\alpha, \beta \in \mathcal{Z}_k(X)$ are *rationally equivalent* if $[\alpha] = [\beta]$ in $\mathrm{CH}_k(X)$, i. e. if $\alpha - \beta \in \mathcal{Z}_k(X)_{\mathrm{rat}}$.

The intersection product that we look for, as motivated in the introduction, can now be rephrased as a bilinear map

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \to \mathrm{CH}^{k+l}(X)$$

that turns $\bigoplus_{k=0}^{\dim X} \mathrm{CH}^k(X)$ into a graded ring.

2.4 Examples

First of all, from the definition of $\mathrm{CH}_k(X)$ it follows that $\mathrm{CH}_k(X) = \mathrm{CH}_k(X_{\mathrm{red}})$. Thus it suffices to consider reduced schemes.

The top Chow group is quite easy to understand:

Example 2.14. If n is the maximum dimension among the irreducible components of X, then $\mathrm{CH}_k(X) = 0$ for $k > n$ and $\mathrm{CH}_n(X) \cong \mathbb{Z}^s$ where s is the number of n-dimensional irreducible components.

The Chow group CH^1 gives back the divisor class group from section 2.1:

Example 2.15. If X is a normal variety of dimension n, we have $\mathrm{CH}_{n-1}(X) = \mathrm{Cl}(X)$.

In particular, for a smooth curve C we get $\mathrm{CH}_0(C) \cong \mathrm{Pic}(C)$ by Theorem 2.4. Note that $\mathrm{Pic}(C)$ fits into a short exact sequence

$$0 \to J(C)(k) \to \mathrm{Pic}(C) \overset{\deg}{\to} \mathbb{Z} \to 0$$

where $J(C)(k)$ are the k-points of the Jacobian of C (a $g(C)$-dimensional abelian variety). Hence, $\mathrm{CH}_0(C)$ is not finitely generated if $g(C) \geq 1$ and k is algebraically closed.

In general, even $\mathrm{CH}_0(X)$ is not easy to understand. If time allows, at the end of the semester we will see some open conjectures on the behaviour of $\mathrm{CH}_0(X)$.

Proposition 2.16. *We have*

$$\mathrm{CH}_k(\mathbb{P}^n) \cong \mathbb{Z}$$

for all $k \in \{0, \dots, n\}$.

More examples can also be found on the first exercise sheet.

In the examples so far, $\mathrm{CH}_k(X)/\ell$ was finite for all ℓ. The following interesting theorem demonstrates that this does not need to be the case:

Theorem 2.17 (Schoen). *Let*

$$E = \{x_0^3 + x_1^3 + x_2^3 = 0\} \subset \mathbb{P}^2_{\mathbb{C}} \, .$$

Then

$$\mathrm{CH}_1(E \times E \times E)/\ell$$

is infinite for all prime numbers ℓ *with* $\ell \equiv 1 \pmod 3$.

This gives a small insight how mysterious the behaviour of Chow groups is in general.

2.5 Proper pushforward

In analogy to homology, we expect $\mathrm{CH}_k(X)$ to be functorial in X, in the sense that a morphism $f\colon X \to Y$ of algebraic schemes should induce a morphism $f_*\colon \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$ of abelian groups such that $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{CH}_k(X)}$ and $(f \circ g)_* = f_* \circ g_*$. Unfortunately, it is impossible to do this for arbitrary morphisms $f\colon X \to Y$, as the following example demonstrates:

Example 2.18. Let $f\colon \operatorname{Spec} k \to \mathbb{A}^1_k$ be the inclusion of a point and let $g\colon \mathbb{A}^1_k \to \operatorname{Spec} k$ be the structure morphism. Then $g \circ f = \mathrm{id}_{\operatorname{Spec} k}$, so we should have $g_* \circ f_* = \mathrm{id}_{\mathrm{CH}_0(\operatorname{Spec} k)}$. However, this is impossible because $\mathrm{CH}_0(\operatorname{Spec} k) \cong \mathbb{Z}$ and $\mathrm{CH}_0(\mathbb{A}^1_k) = 0$, hence g_* and f_* need to be the zero map.

But it turns out that pushforwards can be consistently defined for *proper* morphisms. To this end, we first define a pushforward on cycles:

Definition 2.19. Let $f\colon X \to Y$ be a proper morphism of algebraic schemes. Then $f_*\colon \mathcal{Z}_k(X) \to \mathcal{Z}_k(Y)$ is defined by setting

$$f_* Z = \begin{cases} d \cdot f(Z) & \text{if } [k(Z) : k(f(Z))] = d < \infty \\ 0 & \text{if } [k(Z) : k(f(Z))] = \infty \end{cases}$$

for a k-dimensional subvariety $Z \subset X$, and extending this formula $\mathbb{Z}$-linearly.

Note that $f(Z) \subset Y$ is closed since f is proper, and that $k(Z)$ is a finite extension of $k(f(Z))$ if and only if $f(Z)$ is k-dimensional.

Clearly, we have $(\mathrm{id}_X)_* = \mathrm{id}_{\mathcal{Z}_k(X)}$. Furthermore, we have:

Lemma 2.20. *For proper morphisms* $g\colon W \to X$ *and* $f\colon X \to Y$, $(f \circ g)_* = f_* \circ g_*$.

Proof. Let $Z \subset W$ be a k-dimensional subvariety. If $\dim f(g(Z)) < \dim g(Z)$ or $\dim g(Z) < \dim Z$, then $\dim f(g(Z)) < \dim Z$ and we have $f_* g_* Z = 0 = (f \circ g)_* Z$ by definition. If $\dim f(g(Z)) = \dim g(Z) = \dim Z = k$, we have

$$\begin{aligned} f_* g_* Z &= [k(Z) : k(g(Z))] f_* g(Z) \\ &= [k(g(Z)) : k(f(g(Z)))][k(Z) : k(g(Z))] f(g(Z)) \\ &= [k(Z) : k(f(g(Z))] f(g(Z)) \\ &= (f \circ g)_* Z \end{aligned}$$

by definition and by the multiplicity of degrees in a tower of field extensions. □

The following important theorem is the main result of this section:

Theorem 2.21. *Let $f\colon X \to Y$ be a proper morphism of algebraic schemes. Then we have $f_*\alpha \in \mathcal{Z}_k(Y)_{rat}$ for all $\alpha \in \mathcal{Z}_k(X)_{rat}$.*

Proof. Since $\mathcal{Z}_k(X)_{\text{rat}}$ is generated by $\operatorname{div}(\phi)$ for $(k+1)$-dimensional subvarieties $W \subset X$ and $\phi \in k(W)^*$, we can replace X by such a W and α by $\operatorname{div}(\phi)$. Hence, we want to show $f_* \operatorname{div}(\phi) \in \mathcal{Z}_k(Y)_{\text{rat}}$ for a $(k+1)$-dimensional variety X and for $\phi \in k(X)^*$. By replacing Y with $f(X)$ (note that $f(X) \subset Y$ is closed since f is proper), we may assume that Y is also a variety and that f is surjective. In particular, we have $\dim Y \leq k+1$. If $\dim Y < k$, the result is obvious because $\mathcal{Z}_k(Y) = 0$. There remain two non-trivial cases:

- **Case 1:** $\dim Y = k+1$

  Then $k(X)$ is a finite field extension of $k(Y)$. We claim that $f_* \operatorname{div}(\phi) = \operatorname{div}(N(\phi))$ where $N\colon k(X) \to k(Y)$ is the norm of the field extension $k(Y) \subset k(X)$. In other words, $N(\phi)$ is the determinant of the $k(Y)$-linear map $k(X) \to k(X)$ given by multiplication with ϕ.

  Let $E \subset Y$ be a prime divisor and let $D_1, \ldots, D_m \subset X$ be the prime divisors with $f(D_i) = E$. There are only finitely many of them because they must be irreducible components of $f^{-1}(E) \subsetneq X$. By definition of principal divisors and the pushforward of k-cycles, we need to show

  $$\sum_{i=1}^{m} [k(D_i) : k(E)] \operatorname{ord}_{D_i}(\phi) = \operatorname{ord}_E(N(\phi)) \,.$$

  Note that the subring

  $$B = \mathcal{O}_{X,D_1} \cap \cdots \cap \mathcal{O}_{X,D_m} \subset k(X)$$

  is a finitely generated module over the local ring $A = \mathcal{O}_{Y,E} \subset k(Y)$, whose maximal ideal we denote by $\mathfrak{m}$. Furthermore, B is a 1-dimensional integral domain with fraction field $k(X)$ and its finitely many maximal ideals are $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$, where

$B_{\mathfrak{m}_i} = \mathcal{O}_{X,D_i}$ and $B/\mathfrak{m}_i = k(D_i)$. Since $\operatorname{ord}_{D_i}(\cdot), \operatorname{ord}_E(N(\cdot)) \colon k(X)^* \to \mathbb{Z}$ are group homomorphisms, it suffices to show the following purely algebraic statement for all $b \in B \setminus \{0\}$:

$$\sum_{i=1}^{m} [B/\mathfrak{m}_i : A/\mathfrak{m}] \operatorname{length}_{B_{\mathfrak{m}_i}}(B_{\mathfrak{m}_i}/b) = \operatorname{length}_A(A/N(b))$$

This follows from the following two facts from commutative algebra:

- **Fact 1:** $\operatorname{length}_A(B/b) = \operatorname{length}_A(A/N(b))$ (see [Sta22, Tag 02MI] or [Ful98, Lemma A.2.6])
- **Fact 2:** $\operatorname{length}_A(M) = \sum_{i=1}^{m} [B/\mathfrak{m}_i : A/\mathfrak{m}] \operatorname{length}_{B_{\mathfrak{m}_i}}(M_{\mathfrak{m}_i})$ for any finitely generated A-module M (see [Sta22, Tag 02M0] or [Ful98, Lemma A.2.2, Lemma A.2.3])

- **Case 2:** $\dim Y = k$

  We claim that $f_* \operatorname{div}(\phi) = 0$ in this case. For this we need to show

  $$\sum_{D \subset X} [k(D) : k(Y)] \operatorname{ord}_D(\phi) = 0$$

  where the sum is taken over all prime divisors $D \subset X$ such that $f(D) = Y$ (note that $\operatorname{ord}_D(\phi) = 0$ for all but finitely many D). By replacing X with $X \times_Y \operatorname{Spec} k(Y)$, Y with $\operatorname{Spec} k(Y)$, and finally k with $k(Y)$, we may assume X is a proper curve over $Y = \operatorname{Spec} k$. Let $h \colon \tilde{X} \to X$ be the normalization of X. Note that $k(\tilde{X}) = k(X)$ and thus $h_* \operatorname{div}(\phi) = \operatorname{div}(\phi)$ by Case 1. Since $\tilde{X}$ is a smooth projective curve, there exists a finite map $g \colon \tilde{X} \to \mathbb{P}^1$. Let $p \colon \mathbb{P}^1 \to \operatorname{Spec} k$ be the structure morphism, so $f \circ h = p \circ g$. If we could show $p_* \operatorname{div}(\psi) = 0$ for all $\psi \in k(\mathbb{P}^1)^*$, then using Case 1 for g it would follow by Lemma 2.20 that

  $$f_* \operatorname{div}(\phi) = f_* h_* \operatorname{div}(\phi) = p_* g_* \operatorname{div}(\phi) = p_* \operatorname{div}(N_g(\phi)) = 0\ .$$

  Hence, it remains to consider the case $X = \mathbb{P}^1$ and $Y = \operatorname{Spec} k$.

  Any $\phi \in k(\mathbb{P}^1)^* = k(t)^*$ can be factored into a product of irreducible polynomials in $k[t]$ and their inverses. Thus it suffices to prove $f_* \operatorname{div}(\phi) = 0$ for an irreducible polynomial $\phi \in k[t]$. Let $p \in \mathbb{P}^1 \setminus \{\infty\}$ be the corresponding closed point with residue field $\kappa(p) = k[t]/\phi$. Note that $\phi \in \mathcal{O}_{\mathbb{P}^1,q}$ is a unit for all $q \in \mathbb{P}^1 \setminus \{p, \infty\}$. Since ϕ generates the maximal ideal of $\mathcal{O}_{\mathbb{P}^1,p} = k[t]_{(\phi)}$, we have $\operatorname{ord}_p(\phi) = 1$. Moreover, we have $\mathcal{O}_{\mathbb{P}^1,\infty} = k[t^{-1}]_{(t^{-1})}$ with maximal ideal (t^{-1}). If $\phi \in k[t]$ is a polynomial of degree d, then d is the smallest exponent such that $\left(t^{-1}\right)^d \cdot \phi \in \mathcal{O}_{\mathbb{P}^1,\infty}$. Hence, $\operatorname{ord}_\infty(\phi) = -d$. In total, we get $\operatorname{div}(\phi) = 1 \cdot p - d \cdot \infty$ and thus

  $$f_* \operatorname{div}(\phi) = ([\kappa(p) : k] \cdot 1 - [\kappa(\infty) : k] \cdot d) \cdot \operatorname{Spec} k = (d \cdot 1 - 1 \cdot d) \cdot \operatorname{Spec} k = 0\ . \ \square$$

Remark 2.22. In most places, we just used that proper maps are closed (e. g. in order to even define the pushforward on the level of cycles). However, this cannot be enough, since the maps in Example 2.18 are closed, but not proper. Specifically, Case 2 of the preceding proof used at one place the crucial fact that smooth proper curves are projective. This requires the full strength of properness.

Definition 2.23. Let $f\colon X \to Y$ be a proper morphism of algebraic schemes. Then

$$f_*\colon \mathrm{CH}_k(X) \to \mathrm{CH}_k(Y)$$

is the morphism induced by $f_*\colon \mathcal{Z}_k(X) \to \mathcal{Z}_k(Y)$ via Theorem 2.21.

Now that we have constructed the pushforward, we can define the degree map on the Chow group of 0-cycles:

Definition 2.24. Let X be proper. The *degree map*

$$\deg\colon \mathrm{CH}_0(X) \to \mathbb{Z}$$

is defined as the pushforward $\mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k)$ of the structure morphism $X \to \operatorname{Spec} k$, composed with the canonical isomorphism $\mathrm{CH}_0(\operatorname{Spec} k) \cong \mathbb{Z}$, $[\operatorname{Spec} k] \mapsto 1$.

In other words, the degree of $\sum n_i[p_i] \in \mathrm{CH}_0(X)$ is given by $\sum n_i[\kappa(p_i) : k]$.

Lemma 2.25. *For a proper morphism $f\colon X \to Y$ and $\alpha \in \mathrm{CH}_0(X)$, we have*

$$\deg(f_*\alpha) = \deg(\alpha) \; .$$

Proof. This follows directly from Definition 2.24 and Lemma 2.20. □

In analogy to homology, we also have the following sort of Mayer-Vietoris sequence, arising from the pushforward by closed inclusions.

Lemma 2.26. *If $X_1, X_2 \subset X$ are closed subsets of an algebraic scheme X, the sequence*

$$\mathrm{CH}_k(X_1 \cap X_2) \to \mathrm{CH}_k(X_1) \oplus \mathrm{CH}_k(X_2) \to \mathrm{CH}_k(X_1 \cup X_2) \to 0$$

is exact.

Proof. Exercise. □

Another useful application of proper pushforwards is an alternative characterization of rational equivalence in terms of families over $\mathbb{P}^1$. This will be carried out in detail on the second exercise sheet.

2.6 Flat pullback

While the Chow groups $\mathrm{CH}_k(X)$, graded by the dimension k of cycles, behave similar to homology in the sense that pushforwards exist (at least for proper morphisms), the notation suggests that the Chow groups $\mathrm{CH}^j(X)$, graded by the codimension j, might behave similar to cohomology in the sense that pullbacks exist. In other words, for fixed j we want $X \mapsto \mathrm{CH}^j(X)$ to be a contravariant functor from equidimensional algebraic schemes to abelian groups. This means that there should exist a morphism $f^*\colon \mathrm{CH}^j(Y) \to \mathrm{CH}^j(X)$ for every morphism $f\colon X \to Y$ such that $(\mathrm{id}_X)^* = \mathrm{id}_{\mathrm{CH}^j(X)}$ and $(f \circ g)^* = g^* \circ f^*$. Unfortunately, it is impossible to do this for arbitrary morphisms $f\colon X \to Y$, as the following example demonstrates:

Example 2.27. Let $X = \mathbb{P}^1$ and $Y = (\mathbb{P}^1 \times \{0\}) \cup (\{0\} \times \mathbb{A}^1) \subset \mathbb{P}^1 \times \mathbb{A}^1$ (i. e. $\mathbb{P}^1$ and $\mathbb{A}^1$ glued together at 0). Let $f\colon X \to Y$ be the inclusion of $\mathbb{P}^1 \times \{0\}$ and let $g\colon Y \to X$ be the projection to the first factor. Then $g \circ f = \mathrm{id}_X$, so we would have $f^* \circ g^* = \mathrm{id}_{\mathrm{CH}^1(X)}$ if contravariant pullbacks of f and g exist. But this is a contradiction because $\mathrm{CH}^1(X) \cong \mathbb{Z}$ and $\mathrm{CH}^1(Y) = 0$ (since $\mathrm{CH}^1(\mathbb{P}^1 \times \{0\})$ is generated by $[(0,0)]$ and $\mathrm{CH}^1(\{0\} \times \mathbb{A}^1) = 0$), so g^* and f^* would need to be the zero map.

However, it turns out that pullbacks can be consistently defined for all flat morphisms. Recall that a morphism $f\colon X \to Y$ is *flat* if for all affine open subsets $\operatorname{Spec} B \subset X$ and $\operatorname{Spec} A \subset Y$ such that $f(\operatorname{Spec} B) \subset \operatorname{Spec} A$, the induced ring homomorphism $A \to B$ is flat. This means that for any exact sequence

$$0 \to L \to M \to N \to 0$$

of A-modules, the sequence

$$0 \to L \otimes_A B \to M \otimes_A B \to N \otimes_A B \to 0$$

is exact.

Important examples of flat morphisms are:

- the inclusion of an open subset $U \subset X$
- the projections $X \times Y \to X$ and $X \times Y \to Y$ from a product (over $\operatorname{Spec} k$) of equidimensional schemes
- affine or projective bundles $p\colon E \to X$
- a dominant morphism $X \to C$ from a variety X to a smooth curve C

We will always assume that a flat morphism has constant relative dimension (this is automatic if the morphism maps to a variety).

As for pushforwards, let us first define pullbacks on the level of cycles:

Definition 2.28. Let $f\colon X \to Y$ be a flat morphism of relative dimension r. Then $f^*\colon \mathcal{Z}_k(Y) \to \mathcal{Z}_{k+r}(X)$ is defined by setting

$$f^*Z = c(f^{-1}(Z))$$

for a k-dimensional subvariety $Z \subset Y$, and extending this formula $\mathbb{Z}$-linearly.

Here $f^{-1}(Z) = X \times_Y Z$ denotes the scheme-theoretic inverse image, which is by flatness a purely $(k+r)$-dimensional subscheme of X.

Clearly, we have $(\mathrm{id}_X)^* = \mathrm{id}_{\mathcal{Z}_k(X)}$. In order to prove that the pullback of cycles is functorial, we need the following lemma:

Lemma 2.29. *For any closed subscheme $Z \subset X$ of pure dimension k, we have*

$$f^*c(Z) = c(f^{-1}(Z)) \ .$$

Proof. Let D be an irreducible $(k+r)$-dimensional component of $f^{-1}(Z)$. Let $E = \overline{f(D)} \subset Z$. Let $A = \mathcal{O}_{Z,E}$ and $B = \mathcal{O}_{f^{-1}(Z),D}$. Then $A \to B$ is flat and local. The coefficient of D in $c(f^{-1}(Z))$ is given by $\mathrm{length}_B(B)$, while its coefficient in $f^*c(Z)$ is given by $\mathrm{length}_A(A) \cdot \mathrm{length}_B(B/\mathfrak{m}_A B)$. The statement thus follows from the fact that these two quantities are equal (see [Ful98, Lemma A.4.1]). □

Corollary 2.30. *For flat morphisms $g\colon W \to X$ and $f\colon X \to Y$, $(f \circ g)^* = g^* \circ f^*$.*

Proposition 2.31. *Let $f\colon X \to Y$ be proper and $g\colon Y' \to Y$ be flat. Consider the fibre product $X' = X \times_Y Y'$, the induced proper morphism $f'\colon X' \to Y'$, and the induced flat morphism $g'\colon X' \to X$. Then we have*

$$f'_* \circ g'^* = g^* \circ f_* \ .$$

Proof. Since flatness and properness are stable under base change, it suffices to prove $f'_*g'^*[X] = g^*f_*[X]$ when X is a variety. By replacing Y with $f(X)$, we may also assume that Y is a variety and that f is surjective. Let $f_*[X] = d\cdot[Y]$ for some $d \in \mathbb{Z}_{\geq 0}$. Then it remains to show $f'_*[X'] = d[Y']$ (because $g'^*[X] = [X']$ and $g^*[Y] = [Y']$). The statement is clear if $d = 0$. For $d > 0$, it follows from the fact that $\mathrm{length}_A(M) = d \cdot \mathrm{length}_B(M)$ for a local ring homomorphism $A \to B$ with $d = [B/\mathfrak{m}_B : A/\mathfrak{m}_A]$ and any B-module M (see [Ful98, Lemma A.1.3]). □

Analogously to proper pushforward, the following important result allows us to pass to Chow groups:

Theorem 2.32. *Let $f\colon X \to Y$ be a flat morphism of relative dimension r. Then we have $f^*\alpha \in \mathcal{Z}_{k+r}(X)_{rat}$ for all $\alpha \in \mathcal{Z}_k(Y)_{rat}$.*

Proof. Without loss of generality, let Y be a variety and $\alpha = \operatorname{div}(\phi)$ for some $\phi \in k(Y)^*$. Let $X_1, \dots, X_m$ be the irreducible components of X and $n_i = \operatorname{length}_{\mathcal{O}_{X,X_i}}(\mathcal{O}_{X,X_i})$ their multiplicities. By flatness, the restrictions $f|_{X_i} \colon X_i \to Y$ are dominant, and thus induce field extensions $k(Y) \subset k(X_i)$. We denote the image of ϕ in $k(X_i)$ by $\phi \circ f|_{X_i}$. We claim that

$$f^* \operatorname{div}(\phi) = \sum_{i=1}^{m} n_i \operatorname{div}(\phi \circ f|_{X_i}) \ .$$

To prove this, let $D \subset X$ be a prime divisor. Let $E = \overline{f(D)}$. If $E = Y$, the coefficient of D is 0 on both sides. Otherwise, $E \subset Y$ is a prime divisor. Consider the 1-dimensional local rings $A = \mathcal{O}_{Y,E}$ and $B = \mathcal{O}_{X,D}$, together with the induced morphism $A \to B$, which is flat and local. The components X_i such that $X_i \cap D \neq \emptyset$ precisely correspond to the minimal prime ideals $\mathfrak{q}_i \subset B$, and we have $n_i = \operatorname{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$. The fraction fields of the integral domains A and $B/\mathfrak{q}_i$ are $k(Y)$ and $k(X_i)$, respectively. Now the coefficient of D in $f^* \operatorname{div}(\phi)$ is by definition exactly $\operatorname{ord}_A(\phi) \cdot \operatorname{length}_B(B/\mathfrak{m}_A B)$, while the coefficient of D in $\sum_{i=1}^m n_i \operatorname{div}(\phi \circ f|_{X_i})$ is $\sum_{i=1}^m \operatorname{ord}_{B/\mathfrak{q}_i}(\phi \circ f|_{X_i}) \cdot \operatorname{length}_{B_{\mathfrak{q}_i}}(B_{\mathfrak{q}_i})$. Hence, the statement follows from [Ful98, Lemma A.2.7]. □

Definition 2.33. Let $f \colon X \to Y$ be a flat morphism of relative dimension r. Then

$$f^* \colon \operatorname{CH}_k(Y) \to \operatorname{CH}_{k+r}(X)$$

is the morphism induced by $f^* \colon \mathcal{Z}_k(Y) \to \mathcal{Z}_{k+r}(X)$ via Theorem 2.32.

Example 2.34. If $f \colon X \to Y$ is a flat morphism which is finite of degree d (in particular, f is proper), then $f_* f^* \alpha = d \cdot \alpha$ for all $\alpha \in \operatorname{CH}_k(Y)$.

As a first application of pullbacks, let us state the localization exact sequence, which is a very useful tool for computing Chow groups:

Proposition 2.35. *Let X be an algebraic scheme, let $Y \subset X$ be a closed subscheme, and let $U = X \setminus Y$. Consider the inclusion $i \colon Y \to X$ (which is proper) and the inclusion $j \colon U \to X$ (which is flat of relative dimension 0). Then the sequence*

$$\operatorname{CH}_k(Y) \xrightarrow{i_*} \operatorname{CH}_k(X) \xrightarrow{j^*} \operatorname{CH}_k(U) \to 0$$

is exact.

Proof. Surjectivity of j^* follows immediately by taking the closure in X of k-dimensional subvarieties of U.

Now let $\alpha \in \mathcal{Z}_k(X)$ be a k-cycle such that $j^*[\alpha] = 0$ in $\operatorname{CH}_k(U)$. By definition of $\mathcal{Z}_k(U)_{\text{rat}}$, this implies $j^*\alpha = \sum n_i \operatorname{div}(\phi_i)$ for some $(k+1)$-dimensional subvarieties $W_i \subset U$, non-zero rational functions $\phi_i \in k(W_i)^*$, and integers $n_i \in \mathbb{Z}$. Note that each ϕ_i is also a non-zero rational function on the $(k+1)$-dimensional subvariety $\overline{W_i} \subset X$. Hence, the

k-cycle $\beta = \alpha - \sum n_i \operatorname{div}(\phi_i) \in \mathcal{Z}_k(X)$ satisfies $j^*\beta = 0$ in $\mathcal{Z}_k(U)$. This means that the coefficient of β at any k-dimensional subvariety $Z \subset X$ with $Z \cap U \neq \emptyset$ is 0 (recall that $\mathcal{Z}_k(U)$ is the free abelian group over the k-dimensional subvarieties of U, which bijectively correspond to the k-dimensional subvarieties $Z \subset X$ with $Z \cap U \neq \emptyset$). Hence, β is a $\mathbb{Z}$-linear combination of subvarieties $Z \subset Y$, i. e. $\beta = i_*\gamma$ for some $\gamma \in \mathcal{Z}_k(Y)$. Therefore, $[\alpha] = [\beta] = i_*[\gamma]$ in $\mathrm{CH}_k(X)$, which proves exactness at $\mathrm{CH}_k(X)$. □

For the next proposition, recall that an affine bundle of rank r over X is an algebraic scheme E together with a morphism $p\colon E \to X$ such that every point of X has an open neighbourhood $U \subset X$ where $p^{-1}(U) \cong U \times \mathbb{A}^r$ and the composition $U \times \mathbb{A}^r \xrightarrow{\cong} p^{-1}(U) \xrightarrow{p} U$ agrees with the projection to the first factor. It follows that p is flat of relative dimension r, so we can consider the pullback $p^*\colon \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$.

Proposition 2.36. *Let $p\colon E \to X$ be an affine bundle of rank r. Then*

$$p^*\colon \mathrm{CH}_k(X) \to \mathrm{CH}_{k+r}(E)$$

is surjective.

We will see later that p^* is even an isomorphism if E is a vector bundle.

Proof. Let $\emptyset \neq U \subset X$ be an affine open subset where $p^{-1}(U) \cong U \times \mathbb{A}^r$, and let $Y = X \setminus U$. Consider the diagram

$$\begin{array}{ccccccc}
\mathrm{CH}_k(Y) & \longrightarrow & \mathrm{CH}_k(X) & \longrightarrow & \mathrm{CH}_k(U) & \longrightarrow & 0 \\
\downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \\
\mathrm{CH}_{k+r}(p^{-1}(Y)) & \longrightarrow & \mathrm{CH}_{k+r}(E) & \longrightarrow & \mathrm{CH}_{k+r}(U \times \mathbb{A}^r) & \longrightarrow & 0
\end{array}$$

whose rows are exact by Proposition 2.35 and which commutes by Corollary 2.30 and Proposition 2.31. We may assume inductively that the statement is true for Y because there is no infinite descending chain $X \supsetneq Y \supsetneq Y' \supsetneq \cdots$ of closed subsets. Hence, the first vertical pullback is surjective. If we could show that the last vertical pullback is surjective as well, we would be done by a diagram chase. Therefore, we may assume that X is affine and $E = X \times \mathbb{A}^r$ is the trivial bundle of rank r. Furthermore, we may assume $r = 1$ since the projection $X \times \mathbb{A}^r \to X$ factors through the projections $X \times \mathbb{A}^{i-1} \times \mathbb{A}^1 \to X \times \mathbb{A}^{i-1}$ for $i = 1, \ldots, r$, and each $X \times \mathbb{A}^{i-1}$ is affine as well.

Let $Z \subset X \times \mathbb{A}^1$ be a closed subvariety of dimension $k+1$. By replacing X with $\overline{p(Z)}$, we may assume that $p|_Z\colon Z \to X$ is dominant. Thus either $\dim X = k$ or $\dim X = k+1$. In the first case, it follows that $Z = X \times \mathbb{A}^1$, so we have $[Z] = p^*[X]$ in $\mathrm{CH}_{k+1}(X \times \mathbb{A}^1)$. For the second case, let $X = \operatorname{Spec} A$ and let $\mathfrak{q} \subset A[t]$ be the prime ideal corresponding to $Z \subset X \times \mathbb{A}^1$. Let K be the function field of X (i. e. the fraction field of A). Since $K[t]$ is a principal ideal domain and $Z \subsetneq X \times \mathbb{A}^1$ dominates X, the ideal $\mathfrak{q}K[t]$ (corresponding to the intersection of Z with the fiber $\mathbb{A}^1_K$ at the generic point $\operatorname{Spec} K \subset X$) is generated by a non-zero polynomial $r \in K[t]$. Therefore, we have $\operatorname{div}(r)|_{U \times \mathbb{A}^1} = Z \cap (U \times \mathbb{A}^1)$ for a non-empty open subset $U \subset X$. In other words, $Z - \operatorname{div}(r)$ is a $\mathbb{Z}$-linear combination of $(k+1)$-dimensional subvarieties $W \subset X \times \mathbb{A}^1$ such that $p(W) \subset X \setminus U$. But this means that $Y = \overline{p(W)}$ has dimension k, so we have $[W] = p^*[Y]$ by the first case. □

The statement of the above proposition is already interesting for $k < 0$: It implies $\mathrm{CH}_l(E) = 0$ for all $l < r$. In particular, taking $X = \operatorname{Spec} k$, we obtain $\mathrm{CH}_l(\mathbb{A}^r) = 0$ for all $l < r$.

3 Intersection product

Let X be a smooth variety. Our goal is to define an *intersection product*

$$\mathrm{CH}^k(X) \times \mathrm{CH}^l(X) \to \mathrm{CH}^{k+l}(X)$$

making $\mathrm{CH}^\bullet(X)$ into a graded commutative ring (with $1 = [X]$), and thus relating the different Chow groups with each other. The crucial property we want to have is $[Z_1] \cdot [Z_2] = [Z_1 \cap Z_2]$ whenever $Z_1, Z_2 \subset X$ intersect generically transversally.

There exist (at least) two different ways to define the intersection product:

1. **First approach:** moving lemma

   If we can show that any two cycles $\alpha \in \mathrm{CH}^k(X)$ and $\beta \in \mathrm{CH}^l(X)$ can be represented as $\alpha = \sum m_i[Z_i]$ and $\beta = \sum n_j[W_j]$ such that Z_i and W_j intersect generically transversally for all pairs (i, j), we can try to define $\alpha \cdot \beta$ simply as $\sum m_i n_j [Z_i \cap W_j]$. While such a moving lemma directly implies the uniqueness of the intersection product, it is much harder to prove its existence in this way: The rational equivalence class $\sum m_i n_j [Z_i \cap W_j]$ might depend on the chosen representations $\alpha = \sum m_i[Z_i]$ and $\beta = \sum n_j[W_j]$. Hence, it is unclear if $\alpha \cdot \beta \in \mathrm{CH}^{k+l}(X)$ is well-defined.

   Another disadvantage of this approach is that we have no control over $[Z_1] \cdot [Z_2]$ if $Z_1, Z_2 \subset X$ do not intersect generically transversally. In particular, we do not know if the class $[Z_1] \cdot [Z_2]$ can be represented by a cycle on $Z_1 \cap Z_2$.

2. **Second approach:** explicit formula

   In [Ful98], Fulton follows a different approach: He provides an explicit formula for $[Z_1] \cdot [Z_2]$ inside the Chow group of $Z_1 \cap Z_2$ for *any* two subvarieties $Z_1, Z_2 \subset X$. This solves the problems of the first approach. However, some properties of the intersection product (e. g. associativity) which would be trivial using the moving lemma require a bit more work.

In this lecture, we will follow the second approach.

Another viewpoint on the intersection product is the following: Intersecting with a fixed k-dimensional subvariety $Z \subset X$ gives (provided that we have constructed the intersection product as in the second approach) a well-defined morphism $\mathrm{CH}_l(X) \to \mathrm{CH}_{l-k}(Z)$. This may be regarded as the pullback via the inclusion $Z \hookrightarrow X$. Note that this inclusion is not flat (unless $Z = X$), but it behaves a bit like a flat morphism of relative dimension $-k$. Thus, the definition of the intersection product may be seen as a generalization of flat pullbacks to certain non-flat morphisms.

In fact, it would suffice to define such a pullback for a *smooth* subvariety $Z \subset X$. This is because we need it only for the inclusion $\Delta\colon X \hookrightarrow X \times X$ of the diagonal, due to the following trick: For given $\alpha \in \mathrm{CH}^k(X)$ and $\beta \in \mathrm{CH}^l(X)$, we can consider $\alpha \times \beta \in \mathrm{CH}^{k+l}(X \times X)$ and define $\alpha \cdot \beta = \Delta^*(\alpha \times \beta) \in \mathrm{CH}^{k+l}(X)$.

Our first goal is to define the intersection product $D \cdot \alpha \in \mathrm{CH}_{k-1}(X)$, even for a singular variety X, in the case where $\alpha \in \mathrm{CH}_k(X)$ is a k-cycle (modulo rational equivalence) and D is a *Cartier divisor* on X.

3.1 Recap: Cartier divisors

Let X be a variety. Recall that a Cartier divisor on X is a global section of the quotient $\mathcal{K}_X^*/\mathcal{O}_X^*$, where $\mathcal{K}_X^*$ is the sheaf of non-zero rational functions (i. e. the constant sheaf with value $k(X)^*$) and $\mathcal{O}_X^*$ is the sheaf of invertible regular functions (i. e. $\mathcal{O}_X^*(U)$ is the unit group of $\mathcal{O}_X(U)$).

In other words, a Cartier divisor on X is given by a collection $\{(U_\alpha, f_\alpha)\}_\alpha$ of open subsets $U_\alpha \subset X$ and non-zero rational functions $f_\alpha \in k(X)^*$ such that $\bigcup_\alpha U_\alpha = X$ and $\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ for all α and β. Two such collections represent the same Cartier divisor if and only if they differ by invertible regular functions $f_\alpha \in \mathcal{O}_X^*(U_\alpha)$ on a refined cover $\{U_\alpha\}_\alpha$.

The short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{K}_X^* \to \mathcal{K}_X^*/\mathcal{O}_X^* \to 0$$

induces the following long exact sequence in sheaf cohomology:

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathcal{K}_X^*) \to H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{K}_X^*) \to \cdots$$

Since $\mathcal{K}_X^*$ is the constant sheaf with value $k(X)^*$, we have $H^0(X, \mathcal{K}_X^*) = k(X)^*$ and $H^1(X, \mathcal{K}_X^*) = 0$. Moreover, $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) = \mathrm{CaDiv}(X)$ by definition and $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$. Therefore, we get:

$$0 \to \mathcal{O}_X(X)^* \to k(X)^* \to \mathrm{CaDiv}(X) \to \mathrm{Pic}(X) \to 0$$

Recall that $\mathrm{Pic}(X)$ are the isomorphism classes of line bundles on X (i. e. locally free $\mathcal{O}_X$-modules of rank 1), with group structure given by the tensor product.

The image of a Cartier divisor $D \in \mathrm{CaDiv}(X)$ under the surjection $\mathrm{CaDiv}(X) \to \mathrm{Pic}(X)$ is denoted by $\mathcal{O}_X(D)$. Concretely, if D is represented by a collection $\{(U_\alpha, f_\alpha)\}_\alpha$, then the line bundle $\mathcal{O}_X(D)$ is obtained by gluing together the trivial line bundles over each U_α via the transition functions $\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$.

The image of a non-zero rational function $f \in k(X)^*$ under the map $k(X)^* \to \mathrm{CaDiv}(X)$ is denoted by $\mathrm{div}(f)$ and called the *principal Cartier divisor* of f. Explicitly, $\mathrm{div}(f)$ is represented by the single pair (X, f).

Due to the above exact sequence, $\mathrm{Pic}(X)$ is the quotient of Cartier divisors by principal Cartier divisors.

The *order* $\mathrm{ord}_E(D)$ of a Cartier divisor $D \in \mathrm{CaDiv}(X)$ at a prime divisor $E \subset X$ is given by the order of vanishing (see Definition 2.6) of $D_{\eta_E} \in (\mathcal{K}_X^*/\mathcal{O}_X^*)_{\eta_E} = k(X)^*/\mathcal{O}_{X,E}^*$. Note that the group homomorphism $\mathrm{ord}_E\colon k(X)^* \to \mathbb{Z}$ descends to $k(X)^*/\mathcal{O}_{X,E}^* \to \mathbb{Z}$ because units have order 0. Explicitly, the order of $D = \{(U_\alpha, f_\alpha)\}$ at $E \subset X$ is simply equal to $\mathrm{ord}_E(f_\alpha)$ for any α such that $U_\alpha \cap E \neq \emptyset$. This is well-defined because f_α only differs by a unit of $\mathcal{O}_{X,E}$ for different choices of α.

To a Cartier divisor $D \in \mathrm{CaDiv}(X)$, we can now associate the divisor

$$\sum_{E \subset X} \mathrm{ord}_E(D) \cdot E \ ,$$

where the sum is taken over all prime divisors $E \subset X$ (note that for $D = \{(U_\alpha, f_\alpha)\}$, the associated Weil divisor restricts to the principal divisor $\mathrm{div}(f_\alpha)$ on U_α). This defines a group homomorphism $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$. It follows from the definitions that principial Cartier divisors are sent to principal Weil divisors, so we get an induced group homomorphism $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$.

We recall the following two facts:

Proposition 3.1. *If X is normal, the maps $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ are injective.*

Proposition 3.2. *If X is locally factorial, the maps $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ are isomorphisms.*

In general, $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ do not need to be injective, as the following interesting example demonstrates:

Example 3.3. Consider the cuspidal curve $X = \{y^2 = x^3\} \subset \mathbb{A}^2$. Then $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Cl}(X) = 0$ are surjective, and their kernels are isomorphic to the additive group of the ground field k.

To prove this, the following explicit description of $\mathrm{CaDiv}(X)$ for a (possibly singular) curve X is useful:

Lemma 3.4. *Let X be a curve. Then the map*

$$\mathrm{CaDiv}(X) \to \bigoplus_{p \in X} k(X)^*/\mathcal{O}_{X,p}^*$$

is an isomorphism, which is given by sending a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ to its images in the stalks $(\mathcal{K}_X^*/\mathcal{O}_X^*)_p = k(X)^*/\mathcal{O}_{X,p}^*$ at all closed points $p \in X$.

Proof. First note that the map is well-defined (i. e. it maps to the direct sum, not just the direct product of the stalks) because X is quasi-compact and any non-zero rational function is regular and invertible on a non-empty open subset.

It is clear that the map is injective: If a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$ vanishes on a neighbourhood of each closed point, it already vanishes on X.

To prove surjectivity, let $p \in X$ be a closed point and let $f \in k(X)^*$ be a non-zero rational function. We need to show that there exists a Cartier divisor D whose stalk at p is f and whose stalk at any other closed point $q \neq p$ lies in $\mathcal{O}_{X,q}^*$. For this, take $\emptyset \neq U \subset X$ such that $f \in \mathcal{O}_X^*(U)$. We consider the cover $X = U_1 \cup U_2$ with $U_1 = U \cup \{p\}$ and $U_2 = X \setminus \{p\}$. Note that U_1 is open because X is a curve, so U is the complement of finitely many closed points. Now $D = \{(U_1, f_1), (U_2, f_2)\}$ with $f_1 = f$ and $f_2 = 1$ defines a Cartier divisor because $\frac{f_1}{f_2} = f$ is a unit on $U_1 \cap U_2 = U$. By construction, D has the desired property. □

Remark 3.5. This lemma immediately shows that $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$, and thus also $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$, is an isomorphism for a normal (or equivalently, a smooth) curve X: In this case, $\mathcal{O}_{X,p}$ is a discrete valuation ring for any closed point $p \in X$. Hence, ord_p induces an isomorphism $k(X)^*/\mathcal{O}_{X,p}^* \cong \mathbb{Z}$, so $\mathrm{CaDiv}(X) \cong \bigoplus_{p \in X} \mathbb{Z} = \mathrm{Div}(X)$.

On the exercise sheet, you will see in detail how the statement in Example 3.3 follows from Lemma 3.4.

Here is an example where the maps $\mathrm{CaDiv}(X) \to \mathrm{Div}(X)$ and $\mathrm{Pic}(X) \to \mathrm{Cl}(X)$ are injective but not surjective:

Example 3.6. Consider the cone $X = \{xy = z^2\} \subset \mathbb{A}^3$, which is normal but not locally factorial (the local ring at the singular point $(0,0,0)$ is not a unique factorization domain). Then the line $L = \{x = z = 0\} \subset X$ is a Weil divisor which is not in the image of $\mathrm{CaDiv}(X)$, since it cannot be described by a single equation locally around $(0,0,0)$. In fact, it turns out that $\mathrm{Cl}(X) \cong \mathbb{Z}/2$, generated by L, and $\mathrm{Pic}(X) = 0$.

Note that (a projectivized version of) this example already appeared in the first chapter to demonstrate that on some singular varieties no reasonable intersection product exists. This is not a contradiction to our goal to construct $D \cdot \alpha$ for a Cartier divisor D and a cycle α on an arbitrary variety X, because the divisors that contradict the existence of the intersection product there are not Cartier.

The *support* $|D|$ of a Cartier divisor D is the union of all prime divisors $E \subset X$ such that $D_{\eta_E} \in (\mathcal{K}_X^*/\mathcal{O}_X^*)_{\eta_E}$ does not vanish. Concretely, if $D = \{(U_\alpha, f_\alpha)\}_\alpha$, then $|D|$ is the union of all prime divisors $E \subset X$ such that $f_\alpha \in \mathcal{O}_{X,E}^*$ for (some or all, it does not matter) indices α such that $U_\alpha \cap E \neq \emptyset$. Since X is quasi-compact and any non-zero rational function restricts to an invertible regular function on a non-empty open subset, there exist only finitely many such E.

Note that the support of a Cartier divisor might be larger than the support of the associated Weil divisor. This happens if the order of a rational function $\phi \in k(X)^*$ is zero at a prime divisor $E \subset X$ although $\phi \notin \mathcal{O}_{X,E}^*$. This is only possible if $\mathcal{O}_{X,E}$ is not a discrete valuation ring.

A Cartier divisor is called *effective* if there exists a representation $\{(U_\alpha, f_\alpha)\}_\alpha$ such that $f_\alpha \in \mathcal{O}_X(U_\alpha)$ for all α. An effective Cartier divisor determines a closed subscheme of X that is locally defined by a single equation.

3.2 Pseudo-divisors

Let X be a variety. Recall that we want to define $D \cdot \alpha \in \mathrm{CH}_{k-1}(X)$ for a Cartier divisor D and a k-cycle $\alpha \in \mathcal{Z}_k(X)$. In fact, as for the general intersection product, we want to have $D \cdot \alpha \in \mathrm{CH}_{k-1}(|D| \cap |\alpha|)$, where $|\alpha|$ is the union of the k-dimensional subvarieties appearing with non-zero coefficient in α.

By $\mathbb{Z}$-linearity, it suffices to define $D \cdot Z \in \mathrm{CH}_{k-1}(|D| \cap Z)$ for a k-dimensional subvariety $Z \subset X$. If D would restrict to a Cartier divisor on Z, we could simply define $D \cdot Z$ to be the associated Weil divisor of this restriction. However, this is not always possible. For example, D does not restrict to a Cartier divisor on $|D|$.

In contrast, line bundles can be pulled back via arbitrary morphisms. Therefore, $\mathcal{O}_X(D)$ restricts to a line bundle on Z, which determines a rational equivalence class $D \cdot Z \in \mathrm{CH}_{k-1}(Z)$. However, it is unclear from this definition that $D \cdot Z$ lies in $\mathrm{CH}_{k-1}(|D| \cap Z)$.

To solve these problems, we introduce the notion of *pseudo-divisors*. Pseudo-divisors are a common generalization of Cartier divisors and line bundles. They combine their advantages: Pseudo-divisors have a support (like Cartier divisors) and can be pulled back via arbitrary morphisms (like line bundles).

Definition 3.7. A *pseudo-divisor* on a variety X is a triple $D = (Z, L, s)$ consisting of a closed subset $Z \subset X$, a line bundle $L \in \mathrm{Pic}(X)$, and a nowhere vanishing section s of L on the open subset $X \setminus Z$ (i. e. a trivialization of $L|_{X \setminus Z}$). We call $|D| := Z$ the *support* of D, $\mathcal{O}_X(D) := L$ the *line bundle* of D, and $s_D := s$ the *canonical section* of D.

Cartier divisors and line bundles both yield pseudo-divisors:

Definition 3.8. The pseudo-divisor given by a Cartier divisor D is $(|D|, \mathcal{O}_X(D), s_D)$.

Definition 3.9. The pseudo-divisor given by a line bundle L is $(X, L, 1)$.

Definition 3.10. Let $D = (Z, L, s)$ and $D' = (Z', L', s')$ be pseudo-divisors on X. Then we define

$$D + D' := (Z \cup Z', L \otimes L', s \otimes s')$$

and

$$-D := (Z, L^{-1}, 1/s) \ .$$

Definition 3.11. For a morphism $f\colon X \to Y$ of varieties and a pseudo-divisor $D = (Z, L, s)$ on Y, we define the pseudo-divisor f^*D on X by

$$f^*D := (f^{-1}(Z), f^*L, f^*s) \ .$$

Definition 3.12. A Cartier divisor D is said to *represent* a pseudo-divisor E if $|D| \subset |E|$ and there exists an isomorphism $\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ sending $s_D|_{X \setminus |E|}$ to s_E.

Proposition 3.13. *Any pseudo-divisor E is represented by a Cartier divisor D. The representation is unique if $|E| \neq X$.*

Definition 3.14. Let E be a pseudo-divisor on an n-dimensional variety X. Its associated cycle class $[E] \in \mathrm{CH}_{n-1}(|E|)$ is defined as $[D]$, where D is a Cartier divisor representing E.

This is well-defined because D is unique if $|E| \neq X$ (so we even get a well-defined class in $\mathcal{Z}_{n-1}(|E|)$), and if $|E| = X$ then $[E]$ is the image of $\mathcal{O}_X(E)$ under the map $\mathrm{Pic}(X) \to \mathrm{CH}_{n-1}(X)$.

3.3 Intersection with Cartier divisors

Let X be a variety.

Definition 3.15. For a pseudo-divisor D and a k-cycle $\alpha = \sum n_i Z_i \in \mathcal{Z}_k(X)$ we define

$$D \cdot \alpha = \sum n_i [j_i^* D] \in \mathrm{CH}_{k-1}(|D| \cap |\alpha|)$$

where $j_i\colon Z_i \hookrightarrow X$ denote the inclusions and $|\alpha| = \bigcup Z_i$ denotes the support of α.

The main result of this section will be that the map

$$\begin{aligned} \mathcal{Z}_k(X) &\to \mathrm{CH}_{k-1}(|D|) \\ \alpha &\mapsto D \cdot \alpha \end{aligned}$$

is well-defined modulo rational equivalence, so it will induce a map

$$\mathrm{CH}_k(X) \to \mathrm{CH}_{k-1}(|D|) \ .$$

Before we will be able to show this, we prove some basic properties of the intersection product $D \cdot \alpha$. These statements are analogous to the properties that we later want to have for the intersection of two algebraic cycles.

Lemma 3.16. *If D_1 and D_2 are pseudo-divisors on X and $\alpha \in \mathcal{Z}_k(X)$, then*

$$(D_1 + D_2) \cdot \alpha = D_1 \cdot \alpha + D_2 \cdot \alpha$$

in $\mathrm{CH}_{k-1}((|D_1| \cup |D_2|) \cap |\alpha|)$.

Proof. By linearity, we may assume $\alpha = Z$ for a k-dimensional subvariety $Z \subset X$. Let $j \colon Z \hookrightarrow X$ denote the inclusion of Z into X. By definition, we have $(D_1 + D_2) \cdot Z = [j^*(D_1 + D_2)]$. From the definition of the pullback and the sum of pseudo-divisors we see that $j^*(D_1 + D_2) = j^*D_1 + j^*D_2$. Furthermore, if j^*D_1 and j^*D_2 are represented by Cartier divisors C_1 and C_2, respectively, then $j^*D_1 + j^*D_2$ is represented by the Cartier divisor $C_1 + C_2$. Therefore, we have $[j^*D_1 + j^*D_2] = [j^*D_1] + [j^*D_2]$ in $\mathrm{CH}_{k-1}((|D_1| \cup |D_2|) \cap Z)$. Since $[j^*D_1] = D_1 \cdot Z$ and $[j^*D_2] = D_2 \cdot Z$, the statement follows. □

Lemma 3.17 (Projection formula)**.** *Let $f \colon X \to Y$ be a proper morphism, D a pseudo-divisor on Y, and $\alpha \in \mathcal{Z}_k(X)$. Then*

$$f_*(f^*D \cdot \alpha) = D \cdot f_*\alpha$$

in $\mathrm{CH}_{k-1}(|D| \cap f(|\alpha|))$.

Note that on the left hand side f_* denotes the proper pushforward of Chow groups

$$\mathrm{CH}_{k-1}(f^{-1}(|D|) \cap |\alpha|) \to \mathrm{CH}_{k-1}(|D| \cap f(|\alpha|)) \ ,$$

while on the right hand side f_* denotes the proper pushforward of cycles

$$\mathcal{Z}_k(|\alpha|) \to \mathcal{Z}_k(f(|\alpha|)) \ .$$

Proof. By linearity, we may again assume $\alpha = Z$ for a k-dimensional subvariety $Z \subset X$. Let $W = f(Z) \subset Y$. By definition, we have $f_*Z = d \cdot W$, where $d = 0$ if $\dim W < \dim Z$ and $d = [k(Z) : k(W)] < \infty$ if $\dim W = \dim Z$. We consider the restriction $g\colon Z \to W$ and the inclusions $i\colon Z \hookrightarrow X$ and $j\colon W \hookrightarrow Y$, as shown in the following commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{g} & W \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[f]{} & Y \end{array}$$

By definition,

$$f^*D \cdot Z = [i^*f^*D] = [g^*j^*D]$$

and

$$D \cdot f_*Z = D \cdot dW = d \cdot [j^*D] \ .$$

Therefore, it remains to show

$$g_*[g^*E] = d \cdot [E]$$

in $\mathrm{CH}_{k-1}(|E|)$ for the pseudo-divisor $E = j^*D$ on W.

Let C be a Cartier divisor on W that represents E. Since $g\colon Z \to W$ is surjective, g^*C is a well-defined Cartier divisor on Z, and g^*C represents g^*E. Thus we have $[g^*E] = [g^*C]$ and $[E] = [C]$ (by definition of the cycle associated to a pseudo-divisor). Hence, we want to show

$$g_*[g^*C] = d \cdot [C]$$

for any Cartier divisor C on W.

Locally on W, the Weil divisor associated to C is given by $\mathrm{div}(\phi)$ for some $\phi \in k(W)^*$. Then the Weil divisor associated to g^*C is locally described by $\mathrm{div}(\phi)$ as well, where we regard ϕ as a non-zero rational function on Z via the extension of function fields $k(W) \subset k(Z)$. Now it suffices to show

$$g_* \,\mathrm{div}(\phi) = d \cdot \mathrm{div}(\phi) \ .$$

By the proof of Theorem 2.21, we have $g_* \,\mathrm{div}(\phi) = 0$ if $d = 0$ and $g_* \,\mathrm{div}(\phi) = \mathrm{div}(N(\phi))$ if $d > 0$, where $N\colon k(Z) \to k(W)$ denotes the norm of the field extension $k(W) \subset k(Z)$. For $d = 0$, both sides are zero. For $d > 0$, we have $N(\phi) = \phi^d$ since $\phi \in k(W)$. Thus

$$g_* \,\mathrm{div}(\phi) = \mathrm{div}(N(\phi)) = \mathrm{div}(\phi^d) = d \cdot \mathrm{div}(\phi) \ . \qquad \square$$

Lemma 3.18. *Let $f\colon X \to Y$ be a flat morphism of relative dimension r, D a pseudo-divisor on Y, and $\alpha \in \mathcal{Z}_k(Y)$. Then*

$$f^*D \cdot f^*\alpha = f^*(D \cdot \alpha)$$

in $\mathrm{CH}_{k+r-1}(f^{-1}(|D| \cap |\alpha|))$.

Note that on the left hand side f^* denotes the flat pullback of cycles

$$\mathcal{Z}_k(|\alpha|) \to \mathcal{Z}_{k+r}(f^{-1}(|\alpha|)) \ ,$$

while on the right hand side f^* denotes the flat pullback of Chow groups

$$\mathrm{CH}_{k-1}(|D| \cap |\alpha|) \to \mathrm{CH}_{k+r-1}(f^{-1}(|D| \cap |\alpha|)) \ .$$

Proof. By linearity, we may assume $\alpha = W$ for a k-dimensional subvariety $W \subset Y$. Let $Z_1, \ldots, Z_m \subset X$ be the $(k+r)$-dimensional irreducible components of the inverse image $f^{-1}(W)$ and $n_1, \ldots, n_m \in \mathbb{Z}_{>0}$ their multiplicities. Hence, $f^*W = n_1 Z_1 + \cdots + n_m Z_m$. We consider the inclusion $j\colon W \hookrightarrow Y$, as well as the restrictions $g_l\colon Z_l \to W$ and inclusions $i_l\colon Z_l \hookrightarrow X$ for $l \in \{1, \ldots, m\}$. By definition,

$$f^*D \cdot f^*W = \sum_{l=1}^m n_l [i_l^* f^* D] = \sum_{l=1}^m n_l [g_l^* j^* D]$$

and

$$f^*(D \cdot W) = f^*[j^*D] \ .$$

Let C be a Cartier divisor on W that represents j^*D. Since $g_l\colon Z_l \to W$ is surjective for all $l \in \{1, \ldots, m\}$ (because f is flat), $g_l^* C$ is a well-defined cartier divisor on Z_l representing $g_l^* j^* D$. Therefore, it suffices to show

$$\sum_{l=1}^m n_l [g_l^* C] = f^*[C]$$

for any Cartier divisor C on W.

This statement can be checked locally, so we may assume $[C] = \mathrm{div}(\phi)$ for some $\phi \in k(W)^*$. Then $[g_l^* C] = \mathrm{div}(\phi \circ g_l)$ where $\phi \circ g_l \in k(Z_l)^*$ denotes the image of ϕ under the inclusion of function fields $k(W) \subset k(Z_l)$. Thus we need to prove the formula

$$\sum_{l=1}^m n_l \, \mathrm{div}(\phi \circ g_l) = f^* \, \mathrm{div}(\phi) \ .$$

This is exactly what we have shown in the proof of Theorem 2.32. □

3.4 Chern classes

3.5 Deformation to the normal cone

3.6 Construction of the intersection pairing

3.7 Projection formula

3.8 Correspondences

3.9 Applications

4 More on Chow groups

4.1 Mumford's theorem

4.2 Bloch conjecture

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